Chapter 11
Polynomial Tensor-Based FEM for Parametric Reduced-Order Modelling of Nonlinear Structures via Invariant Manifolds

Tiago S. Martins and Daniel J. Rixen

Abstract Reduced-order modelling (ROM) of nonlinear dynamical systems using invariant manifolds has emerged as a powerful tool for capturing the essential dynamics of complex structures while drastically reducing computational cost. A critical step in this approach is the accurate and efficient computation of nonlinear forces from the full-order finite element (FE) model. This work presents a parametric nonlinear FE framework designed for the systematic assembly and extraction of polynomial nonlinear forces, essential for constructing ROMs via invariant manifolds. The framework supports parametric topology and material definitions, enabling flexible modelling of geometrically nonlinear structures. For instance, employing the Green-Lagrange strain measure, the first Piola-Kirchhoff stress, and the St. Venant-Kirchhoff constitutive relation yields quadratic and cubic nonlinear force terms, which are automatically extracted in tensor form—the tensors are also separated by parameter degree. The proposed software architecture facilitates seamless integration with model order reduction techniques, ensuring high-fidelity simulations while maintaining computational efficiency. Applications in structural dynamics, including large-deformation steady-state vibration problems, are possible uses for this approach.

Keywords NL-FEM, SSM, ROM, Polynomial Tensor Extraction

Introduction

The parametrization method for invariant manifolds, initially developed by Cabré et al. [1–3] and subsequently refined by Haro et al. [4], provides a unified framework that subsumes both classical invariant manifold and normal form approaches [5, 6] through the solution of invariance equations in either graph or normal-form styles. This methodology has been successfully extended to damped vibratory systems [7]. Further developments have incorporated generic forcing conditions, parametric excitations, non-polynomial nonlinearities, and piecewise smooth dynamics [8–10]. Computational implementations enable automated reduction for large-scale finite element models through tools such as MORFE2.0 [11] and SSMtool 2.6 [12].

Within parametric reduction frameworks, Marconi et al. [13] decomposed structural displacements into defect-induced and motion-induced components, expressing internal forces as polynomials in both fields through Neumann-expanded strain approximations. More recently, Morsy and Tiso [14] employed polynomial chaos expansions to quantify uncertainty propagation in joint dynamics, demonstrating efficient characterization of parameter-dependent damping and stiffness variations. Building upon these parametric methodologies, the present work considers a power series expansion of the inverse determinant arising in geometry morphing operations.

Regarding the formulation of the full-order model, the state of the art in tensor-based finite element methods, as implemented in the MFEM library [15], employs a partial assembly paradigm to enable high-performance, matrix-free operator evaluation—a critical capability for high-order discretisations on GPU architectures. This approach decomposes the finite element operator into a sequence of variational restrictions (such as for the subdomains and elements) acting on a point-wise operator at quadrature points (relative to the strong form of the PDE), leveraging the tensor-product structure of quadrilateral and hexahedral elements to perform basis evaluations without explicit matrix storage. Partial assembly minimises memory transfer and achieves near-optimal amount of floating point operations by storing only the operator at quadrature points, while

Tiago S. Martins · Daniel J. Rixen
Technical University of Munich, Chair of Applied Mechanics, TUM School of Engineering and Design, Department of Mechanical Engineering. Boltzmannstr. 15 85748 Garching, Germany.
e-mail: tiago.martins@tum.de; rixen@tum.de

Kristin B. Zimmerman, PhD (eds.), *Nonlinear Structures & Systems, Vol. 1*, Conference Proceedings of the Society for Experimental Mechanics Series, https://doi.org/10.1007/978-8-743-81440-5_11

the parallel decomposition naturally separates communication in parallel processes, unstructured mesh topology, structured linear algebra, and point-wise physics.

Parametrisation Method for Reduced Order Modelling

The full-order system (FOM) takes the form of the autonomous first-order in time DAE: $\mathbf{B}\dot{\boldsymbol{y}} = \mathbf{A}(\boldsymbol{y})$. When external excitations are introduced, the system becomes non-autonomous (time-dependent); however, constructing an equivalent autonomous representation is achievable by appending auxiliary equations that govern the forcing terms. The FOM is of dimension N and we seek to derive a ROM of dimension $n \ll N$ with governing ODE $\dot{\boldsymbol{z}} = \mathbf{f}(\boldsymbol{z})$. The ROM must preserve the essential dynamics of the high-dimensional system near a stable equilibrium point, assumed without loss of generality to be at the origin. The ROM trajectories map to the full system's phase space via the manifold parametrisation mapping $\boldsymbol{y} = \mathbf{W}(\boldsymbol{z})$, ensuring accurate representation of the system's behavior. Substituting $\boldsymbol{y} = \mathbf{W}(\boldsymbol{z})$ and $\dot{\boldsymbol{z}} = \mathbf{f}(\boldsymbol{z})$ into $\mathbf{B}\dot{\boldsymbol{y}} = \mathbf{A}(\boldsymbol{y})$ gives rise to the invariance equation

$$\mathbf{B}\nabla\mathbf{W}(\boldsymbol{z})\,\mathbf{f}(\boldsymbol{z}) = \mathbf{A}(\mathbf{W}(\mathbf{z}))\,, \tag{1}$$

which cannot directly be solved for $\mathbf{W}$ and $\mathbf{f}$, the ROM unknowns. To overcome this issue, assume that the FOM dynamics $\mathbf{A}(\boldsymbol{y})$, the ROM dynamics $\mathbf{f}(\boldsymbol{z})$, and the manifold mapping $\mathbf{W}(\boldsymbol{z})$ can all be expressed as power series:

$$\mathbf{A}(\boldsymbol{y}) = \sum_{\boldsymbol{\alpha}\in\mathcal{I}_N} \mathbf{A}_{\boldsymbol{\alpha}}\, \boldsymbol{y}^{\boldsymbol{\alpha}}\,, \qquad \mathbf{W}(\boldsymbol{z}) = \sum_{\boldsymbol{\sigma}\in\mathcal{I}_n} \mathbf{W}_{\boldsymbol{\sigma}}\, \boldsymbol{z}^{\boldsymbol{\sigma}}\,, \qquad \mathbf{f}(\boldsymbol{z}) = \sum_{\boldsymbol{\nu}\in\mathcal{I}_n} \mathbf{f}_{\boldsymbol{\nu}}\, \boldsymbol{z}^{\boldsymbol{\nu}}\,, \tag{2}$$

where we define the set of multi-indices $\mathcal{I}_N = \mathbb{N}_0^N \setminus \{\mathbf{0}\}$ for the multi-index notation yielding $\boldsymbol{y}^{\boldsymbol{\alpha}} = y_1^{\alpha_1} \cdots y_N^{\alpha_N}$, which is monomial of order $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_N$.

Inserting the power series of equation (2) into the invariance equation (1), yields an overall equality of power series. For each multi-index $\mathbf{p} = (p_1, \ldots, p_n)$, equating the coefficients associated with the monomial $\boldsymbol{z}^{\mathbf{p}}$ generates the corresponding homological equation: $[\mathbf{B}\nabla\mathbf{W}(\boldsymbol{z})\,\mathbf{f}(\boldsymbol{z})]|_{\mathbf{p}} = \mathbf{A}(\mathbf{W}(\mathbf{z}))|_{\mathbf{p}}$ which becomes

$$\mathbf{B}\sum_{\boldsymbol{\nu}>\mathbf{0}}^{\mathbf{p}} \sum_{j=1}^{n} \mathbf{W}_{\mathbf{p}-\boldsymbol{\nu}+\mathbf{e}_j}\, f_{\boldsymbol{\nu},j}\,(p_j - \nu_j + 1) = \sum_{\substack{\boldsymbol{\alpha}\in\mathcal{I}_N \\ |\boldsymbol{\alpha}|\le|\mathbf{p}|}} \mathbf{A}_{\boldsymbol{\alpha}}\, \Xi_{\mathbf{p},\boldsymbol{\alpha}} \tag{3}$$

where the $\Xi_{\mathbf{p},\boldsymbol{\alpha}}$ are scalars described by the equation

$$\boldsymbol{\Xi}_{\mathbf{p},\boldsymbol{\alpha}} = \sum_{\substack{\boldsymbol{\sigma}_l\in\mathcal{I}_n^{|\boldsymbol{\alpha}|} \\ \sum\boldsymbol{\sigma}_l=\mathbf{p}}} \left(\prod_{m=1}^{\alpha_1} W_{\boldsymbol{\sigma}_m,1}\right) \cdots \left(\prod_{m=|\boldsymbol{\alpha}|-\alpha_N+1}^{|\boldsymbol{\alpha}|} W_{\boldsymbol{\sigma}_m,N}\right).$$

Most importantly, equation (3) relates coefficients of $\mathbf{W}$ and $\mathbf{f}$ of order $\le |\mathbf{p}|$. In other words, the coefficients $\mathbf{W}_{\mathbf{p}}$ and $\mathbf{f}_{\mathbf{p}}$ are independent from the coefficients of order larger than $|\mathbf{p}|$, which allows for an efficient construction of the ROM via recurrence with increasing order.

For first-order monomials $|\mathbf{p}| = 1$, the invariance equation reduces to the spectral problem:

$$\mathbf{B}\mathbf{W}_1\mathbf{f}_1 = \mathbf{A}_1\mathbf{W}_1, \tag{4}$$

with matrices defined as:

$$\mathbf{A}_1 = [\mathbf{A}_{\mathbf{e}_1} \mid \cdots \mid \mathbf{A}_{\mathbf{e}_N}]_{N\times N}\,, \qquad \mathbf{W}_1 = [\mathbf{W}_{\mathbf{e}_1} \mid \cdots \mid \mathbf{W}_{\mathbf{e}_n}]_{N\times n}\,, \qquad \mathbf{f}_1 = [\mathbf{f}_{\mathbf{e}_1} \mid \cdots \mid \mathbf{f}_{\mathbf{e}_n}]_{n\times n}\,.$$

This equation is solved via Jordan decomposition of the master eigenmodes. The master eigenvalues $\boldsymbol{\lambda}$ are extracted from the diagonal of $\mathbf{f}_1$, and the generalised eigenvector matrix $\boldsymbol{\Phi}$ is correspondingly redefined.

For higher order monomials $|\mathbf{p}| \ge 2$, the parametrisation method yields linear systems of the form:

$$(s_{\mathbf{p}}\mathbf{B} - \mathbf{A}_1)\mathbf{W}_{\mathbf{p}} + \mathbf{B}\boldsymbol{\Phi}\mathbf{f}_{\mathbf{p}} = \mathbf{R}_{\mathbf{p}}, \tag{5}$$

where $s_{\mathbf{p}} = \mathbf{p} \cdot \boldsymbol{\lambda}$ is the superharmonic frequency and $\mathbf{R}_{\mathbf{p}}$ contains previously computed terms (preceding monomials in graded lexicographic order). This underdetermined system ($N + n$ unknowns, N equations) is typically resolved by setting $\mathbf{f}_{\mathbf{p}} = \mathbf{0}$.

Nonlinear resonance occurs when $s_{\mathbf{p}} \approx \lambda_r$, causing ill-conditioning as $s_{\mathbf{p}}\mathbf{B} - \mathbf{A}_1$ nears singularity. Solvability requires $\mathbf{R}_{\mathbf{p}}$ to lie in the combined images of $s_{\mathbf{p}}\mathbf{B} - \mathbf{A}_1$ and $\mathbf{B}\boldsymbol{\Phi}$, leading to three cases: (i) solvable with $\mathbf{f}_{\mathbf{p}} = \mathbf{0}$ if $\mathbf{R}_{\mathbf{p}}$ is orthogonal to resonant left eigenvectors; (ii) solvable with $\mathbf{f}_{\mathbf{p},\mathcal{R}} \neq \mathbf{0}$ if resonance occurs with included modes—inner resonance; (iii) unsolvable if resonance occurs with excluded modes—outer resonance.

Here, $\mathcal{R} \subset \{1, \ldots, n\}$ marks resonant entries r that satisfy $s_{\mathbf{p}} \approx \lambda_r$.

For cases (i) and (ii), uniqueness is enforced via $\mathbf{B}$-orthogonality conditions between $\mathbf{W}_{\mathbf{p}}$ and resonant left eigenvectors. The resonant set $\mathcal{R}$ is determined using *near-resonance* criteria to account for frequency shifts at high amplitudes and for numerical robustness. Two parametrisation styles emerge: normal-form style uses finite threshold for eigenvalue proximity; and graph style treats all inner modes as resonant.

Practical matters

The computational procedure involves handling large matrices whose dimensions grow rapidly with both the reduction order and the number of degrees of freedom in the system. To mitigate these scaling challenges, it is essential to exploit matrix sparsity patterns and symmetric properties while implementing memory-efficient allocation strategies and algorithms with favourable computational complexity.

This methodology employs local polynomial expansions, which imposes inherent limitations on its applicability to nonlinear problems. The approximations remain valid primarily in neighbourhoods of equilibrium points, and extrapolation to distant regions requires careful validation of the solution fidelity. The choice of parametrisation style presents a fundamental trade-off: graph style formulations, while computationally more expensive, prove advantageous for centre manifold computations, whereas normal-form style parametrisations yield simpler reduced dynamics and provide unique characterization of manifold folding behaviour [8]. A critical requirement for avoiding outer resonance phenomena is the comprehensive inclusion of all full-order model modes that exhibit nonlinear resonance with the master modes up to the specified maximum expansion order at the initialization stage.

The case of structural dynamics

This work addresses vibration problems in structural dynamics with parametrised geometry, where a parameter θ controls the nodal coordinates in the finite element model. To parametrise the invariant manifolds, the second-order equations of motion are reformulated as a first-order differential-algebraic system:

$$\mathbf{M}(\theta)\dot{\mathbf{V}} = -\mathbf{K}(\theta)\mathbf{U} - \mathbf{C}(\theta)\mathbf{V} - \mathbf{F}^{\mathrm{nl}}(\theta, \mathbf{U}) + \mathbf{F}^{\mathrm{ext}}(\theta, \boldsymbol{r})\,, \qquad \dot{\mathbf{U}} = \mathbf{V}\,, \qquad \dot{\boldsymbol{r}} = \mathbf{E}(\boldsymbol{r})\,, \qquad \dot{\theta} = 0\,. \tag{6}$$

Here, $\mathbf{U}$ and $\mathbf{V}$ denote the generalised displacement and velocity vectors, respectively. The system matrices—mass $\mathbf{M}(\theta)$, stiffness $\mathbf{K}(\theta)$, and damping $\mathbf{C}(\theta)$—along with the polynomial nonlinear force $\mathbf{F}^{\mathrm{nl}}(\theta, \mathbf{U})$ and external force $\mathbf{F}^{\mathrm{ext}}(\theta, \boldsymbol{r})$, are functions of the geometric parameter θ. Furthermore, forcing is incorporated via an autonomous subsystem $\dot{\boldsymbol{r}} = \mathbf{E}(\boldsymbol{r})$, which describes the evolution of the forcing state $\boldsymbol{r}$.

In the reduced-order model (ROM), the forcing state $\mathbf{r}$ is embedded within the reduced coordinates as $\mathbf{z} = (\widetilde{\mathbf{z}}, \mathbf{r})$. This leads to a direct parametrization of the forcing: $\mathbf{W}^{\mathrm{r}}(\widetilde{\mathbf{z}}, \mathbf{r}) = \mathbf{r}$, and its dynamics are projected as $\dot{\mathbf{r}} = \mathbf{f}^{\mathrm{r}}(\widetilde{\mathbf{z}}, \mathbf{r}) = \mathbf{E}(\mathbf{r})$. Note that this approach requires the forcing to be representable by a finite-dimensional linear state-space model, thus excluding non-smooth periodic functions and stochastic processes.

The second-order nature of the original system imposes a compatibility condition between the displacement and velocity mappings: $\nabla\mathbf{W}^{u}(\boldsymbol{z}), \mathbf{f}(\boldsymbol{z}) = \mathbf{W}^{\mathrm{v}}(\boldsymbol{z})$. Consequently, the homological equation for displacement reduces to:

$$\sum_{\boldsymbol{\nu}>\mathbf{0}}^{\mathbf{p}} \sum_{j=1}^{n} \mathbf{W}^{u}_{\mathbf{p}-\boldsymbol{\nu}+\mathbf{e}j} f_{\boldsymbol{\nu},j}(p_j - \nu_j + 1) = \mathbf{W}^{\mathrm{v}}_{\mathbf{p}},. \tag{7}$$

This relation can be forward-substituted into the momentum balance invariance equation to streamline computations. The detailed derivation of the resulting equation falls beyond the scope of this work—see [8] for details.

Nonlinear Solid Mechanics

This work employs a Lagrangian description of motion, tracking material points from a fixed undeformed configuration V. While physical laws governing equilibrium and tractions are inherently defined in the deformed, current configuration, the Total Lagrangian (TL) formulation enables a consistent framework by referencing all quantities to V. This is achieved through the use of work-conjugate stress and strain measures.

The fundamental challenge is reconciling undeformed-configuration kinematics with current-configuration physics. Two primary approaches address this: 1. the Total Lagrangian (TL) formulation references all quantities to the undeformed configuration V. It employs the Green-Lagrange strain tensor, which permits polynomial expansions in terms of the displacement gradient—a key property exploited in this work; and 2. the Updated Lagrangian (UL) formulation incrementally updates the reference state to the last computed configuration. It typically uses incremental or logarithmic strain measures, where the sum of incremental strains approximates the integrated strain rate. The TL formulation is computationally efficient for hyperelasticity, large rotations, and path-independent materials, as it avoids the cost of frequent spatial frame updates. Its ability to express strain as a polynomial is essential for the parametrisation of invariant manifolds ROM technique. However, TL is unsuitable for phenomena requiring explicit tracking of the current configuration, such as large-sliding contact, fatigue damage, and finite strain plasticity; these are beyond the scope of this work and typically necessitate a UL approach.

To enforce equilibrium in the undeformed configuration, the TL formulation utilises the first Piola-Kirchhoff stress tensor, $\mathbf{P}$. This tensor is defined as the force in the *current* configuration acting per unit area in the *undeformed* configuration. Its traction relation, $\mathbf{t} = \mathbf{P}\mathbf{n}$, gives the current traction vector on a surface with original unit normal $\mathbf{n}$, thereby maintaining physical fidelity to the true force equilibrium. The linear momentum balance in the interior of the undeformed domain V, expressed in terms of the displacement field $\mathbf{u}$, yields the strong form of the equation of motion:

$$\rho\ddot{\mathbf{u}} = \nabla \cdot \mathbf{P} + \mathbf{b}, \tag{8}$$

where ρ is the density, $\ddot{\blacksquare}$ denotes the second material time derivative, ∇ is the gradient with respect to undeformed position coordinates $\mathbf{x}$, and $\mathbf{b}$ is the *present* body force per unit undeformed volume. The boundary ∂V is partitioned into a Dirichlet part S_{u}, where $\mathbf{u} = \bar{\mathbf{u}}$, and a Neumann part S_{t}, where $\mathbf{P}\mathbf{n} = \mathbf{t}$, with $S_{\mathrm{u}} \cap S_{\mathrm{t}} = \emptyset$.

The weak form is derived by multiplying the strong form by a kinematically admissible test function $\delta\mathbf{u}$ (vanishing on S_{u}) and integrating over V. Application of integration by parts and the divergence theorem, incorporating the prescribed traction boundary condition, yields the following:

$$\int_V \rho\,\delta\mathbf{u}^\top \ddot{\mathbf{u}}\,\mathrm{d}V + \int_V \nabla\delta\mathbf{u}^\top\!: \mathbf{P}\,\mathrm{d}V = \int_V \delta\mathbf{u}^\top \mathbf{b}\,\mathrm{d}V + \int_{S_{\mathrm{t}}} \delta\mathbf{u}^\top \mathbf{t}\,\mathrm{d}S. \tag{9}$$

Here, the colon denotes the tensor double contraction ($\mathbf{A}^\top\!: \mathbf{B} = \mathrm{tr}(\mathbf{A}\mathbf{B})$) and $\nabla\delta\mathbf{u}$ is the gradient of the virtual displacement w.r.t. to the undeformed positions $\mathbf{x}$. This weak form serves as the basis for the finite element discretization.

Applying the parametrisation method for reduced-order modelling requires the underlying dynamical system to exhibit polynomial nonlinearity. In the context of hyperelasticity, this is achieved by selecting a strain energy density function that is a polynomial in the components of the strain tensor. For incompressible materials, models such as neo-Hookean and Mooney-Rivlin naturally express strain energy as a polynomial in the invariants of the left Cauchy-Green tensor $\mathbf{F}\mathbf{F}^T$, where $\mathbf{F} = \mathbf{I} + \mathbf{u}$ is the deformation gradient. For compressible materials, however, the dependence on $J = \det(\mathbf{F})$ often introduces non-polynomial terms (e.g., $\ln J$). A common remedy is to expand the volumetric part of the strain energy as a Taylor series around $J = 1$, yielding a polynomial approximation valid for moderate volumetric deformations $(0 < J < 2)$, as $J = 0$ marks singularity.

This work adopts a hyperelastic constitutive law that inherently provides polynomial nonlinearity. It posits a linear relationship between the second Piola-Kirchhoff stress $\mathbf{S}$ and the Green-Lagrange strain tensor $\mathbf{E} = \frac{1}{2}\left(\mathbf{F}^\top\mathbf{F} - \mathbf{I}\right)$:

$$\mathbf{S} = \mathcal{A} : \mathbf{E},$$

where $\mathcal{A}$ is the fourth-order elasticity tensor. The first Piola-Kirchhoff stress, $\mathbf{P}$, is then obtained via the transformation $\mathbf{P} = \mathbf{F}\mathbf{S}$. In the case of a Saint Venant-Kirchhoff material, the stress-strain relation further simplifies to $\mathbf{S} = 2\mu\mathbf{E} + \lambda\,\mathrm{tr}(\mathbf{E})\mathbf{I}$, where $\mathrm{tr}(\blacksquare)$ is the trace operator, and λ and μ are the Lamé constants.

This formulation captures geometric nonlinearity through the quadratic term in $\mathbf{E}$. For small deformation gradients, this term is neglected, recovering the linear infinitesimal strain tensor. Under finite deformations, this term must be retained, introducing higher-order elastic forces.

Substituting the linear hyperelasticity constitutive relation into the principle of virtual work, Eq. (9), reveals a polynomial structure in the displacement $\mathbf{u}$ and displacement gradient $\nabla\mathbf{u}$, which decomposes into linear, quadratic, and cubic terms:

$$
\begin{aligned}
&\int_V \rho\, \delta\mathbf{u}^\top \ddot{\mathbf{u}}\, \mathrm{d}V + \int_V (\nabla\delta\mathbf{u})_{\mathrm{sym}} : \mathcal{A} : (\nabla\mathbf{u})_{\mathrm{sym}}\, \mathrm{d}V \qquad \text{(Linear)} \\
&+ \int_V ((\nabla\mathbf{u})^\top \nabla\delta\mathbf{u})_{\mathrm{sym}} : \mathcal{A} : (\nabla\mathbf{u})_{\mathrm{sym}} + \frac{1}{2}(\nabla\delta\mathbf{u})_{\mathrm{sym}} : \mathcal{A} : ((\nabla\mathbf{u})^\top \nabla\mathbf{u})_{\mathrm{sym}} \mathrm{d}V \qquad \text{(Quadratic)} \\
&+ \int_V \frac{1}{2}(\nabla\delta\mathbf{u}\nabla\mathbf{u})_{\mathrm{sym}} : \mathcal{A} : ((\nabla\mathbf{u})^\top \nabla\mathbf{u})_{\mathrm{sym}}\, \mathrm{d}V \qquad \text{(Cubic)} \\
&= \int_V \delta\mathbf{u}^\top \mathbf{b}\, \mathrm{d}V + \int_{S_\mathrm{t}} \delta\mathbf{u}^\top \mathbf{t}\, \mathrm{d}S_\mathrm{t} \qquad \text{(External)},
\end{aligned}
\tag{10}
$$

where $(\blacksquare)_{\mathrm{sym}} = (\blacksquare + \blacksquare^\top)/2$ denotes the symmetric part of a matrix. Linear damping is incorporated by adding a velocity-proportional term to the left-hand side.

Structures with Parametric Geometry

This work addresses parametric model reduction for structures whose geometry is defined by a parameter. In the finite element method, these parameters control nodal coordinates, thereby inducing a variation in the computational domain. Consider the following three methods for parametrising geometry.

Deformation with Stress Relaxation

The body is subjected to a prescribed deformation, followed by a relaxation step that resets the internal stress state to zero. This process modifies the structural shape while conserving total mass, resulting in a redistribution of material density. The parameter governs the deformation field and the subsequent relaxation forces that eliminate the resulting pre-stress. This introduces additional explicit parametric force terms in the equation of motion.

Pre-stressed Deformation

The body is deformed while preserving the internal stress state, conserving mass via density redistribution. The parameter characterises the applied pre-stress force field that defines a new equilibrium configuration. Equivalently, the parameter may define the pre-strained deformation state directly. This introduces additional explicit parametric forces in the equation of motion.

Direct Modification of the Undeformed Configuration

The intrinsic geometry of the undeformed body is altered while preserving constitutive material properties. We require a continuously differentiable homeomorphism to that describes how the geometry changes with the parameter. Hence, the parameter directly modifies the integration domain in the weak formulation of the equations of motion. In the case of FEM, the parameter controls nodal positions in the undeformed configuration without inducing pre-stress, altering density, or introducing explicit parametric forces. This work focuses on this approach, which presents a greater mathematical and computational challenge. Mathematically, one has

$$
\mathbf{x} = \mathbf{x}(\mu, \mathbf{x}_0) \qquad \text{and} \qquad \mathbf{J} = \nabla_0 \mathbf{x}\,,
$$

where $\mathbf{x}$ is the position in the undeformed geometry that depends on the position in the reference $\mathbf{x}_0$ and depends polynomially on the geometry parameter μ. The jacobian of the undefomed positions w.r.t the reference positions is $\mathbf{J}$, its adjugate is $\operatorname{adj}\mathbf{J}$ and its determinant is $\det\mathbf{J}$—all polynomial on μ. The volume integral and the gradient become

$$
\int_V \blacksquare\, \mathrm{d}V = \int_{V_0} \blacksquare\, \det\mathbf{J}\, \mathrm{d}V_0 \qquad \text{and} \qquad \nabla\blacksquare = \nabla_0\blacksquare \cdot \mathbf{J}^{-1} = \nabla_0\blacksquare \cdot \operatorname{adj}\mathbf{J}\, (\det\mathbf{J})^{-1}\,.
$$

Three principal approaches for handling the inverse deformation gradient are identified. The first introduces an auxiliary scalar field w, constrained via $w \cdot \det\mathbf{J} = 1$, which is enforced in a weak form; this variational formulation increases the system's dimensionality by one (one new degree of freedom at every FE node) and its accuracy is contingent upon discretization choices [16]. A second approach employs a Neumann series expansion of $\mathbf{J}^{-1}$ in the parameter μ, whose accuracy is limited in practice by the truncation order of the asymptotic expansion used in model reduction [13]. We use the third technique that expresses the inverse determinant $(\det\mathbf{J})^{-1}$ itself as a power series in μ, valid within a specific range, with series coefficients given by a combinatorial sum over integer partitions.

Matrix Polynomial Expansions

In the context of parametric FEM, consider the jacobian of the position as a matrix-valued affine function on a single parameter μ, as follows $\nabla_0 \mathbf{x} = \mathbf{J} = \mathbf{J}_0 + \mu \mathbf{J}_1$, where $\mathbf{J}_0$ and $\mathbf{J}_1$ are constant matrices representing different physical contributions to the system. The computational implementation provides efficient methods for expanding the determinant, adjugate and inverse determinant of such parametric matrices as polynomials (or power series) in μ, which is particularly valuable for the ROM parametrisation method.

For the two-dimensional case with 2×2 matrices, the mathematical formulation is straightforward. The determinant expansion follows the quadratic polynomial form $\det \mathbf{J} = c_0 + c_1\mu + c_2\mu^2$, where the coefficients are computed through explicit formulas: $c_0 = \det \mathbf{J}_0$ represents the base system determinant, $c_2 = \det \mathbf{J}_1$ captures the purely parametric contribution, and $c_1 = \mathbf{J}_0^\top : \operatorname{adj} \mathbf{J}_1$ provides the coupling term between the base and parametric components. Simultaneously, the adjugate matrix exhibits a linear dependence $\operatorname{adj} \mathbf{J} = \mathbf{C}_0 + \mathbf{C}_1\mu$, with $\mathbf{C}_0 = \operatorname{adj} \mathbf{J}_0$ and $\mathbf{C}_1 = \operatorname{adj} \mathbf{J}_1$ following directly from the linearity of the adjugate operation for 2×2 matrices. Additionally, we express $(\det \mathbf{J})^{-1}$ as a power series in μ, given that $0 < \det \mathbf{J} < 2c_0$, as follows:

$$(\det \mathbf{J})^{-1} = \frac{1/c_0}{1 - [c_0 - \det \mathbf{J}]/c_0} = \frac{1}{c_0} \sum_{\kappa=0}^{\infty} \left(\frac{1}{c_0}\right)^{\kappa} (c_0 - \det \mathbf{J})^{\kappa} = \sum_{\kappa=0}^{\infty} a_\kappa \mu^\kappa \,,$$

where the polynomial coefficients are as follows

$$a_\kappa = \frac{1}{c_0} \sum_{\substack{\sigma_1,\sigma_2 \in \mathbb{N}_0 \\ \sigma_1 + 2\sigma_2 = \kappa}} \frac{(\sigma_1+\sigma_2)!}{\sigma_1!\,\sigma_2!} \left(\frac{-c_1}{c_0}\right)^{\sigma_1} \left(\frac{-c_2}{c_0}\right)^{\sigma_2} .$$

The three-dimensional case with 3×3 matrices requires more sophisticated mathematical treatment. The determinant expands as a cubic polynomial $\det \mathbf{J} = c_0 + c_1\mu + c_2\mu^2 + c_3\mu^3$, while the adjugate matrix follows a quadratic polynomial $\operatorname{adj} \mathbf{J}(\mu) = \mathbf{C}_0 + \mathbf{C}_1\mu + \mathbf{C}_2\mu^2$. The coefficients are derived using invariant theory and properties of the characteristic polynomial. The constant terms $c_0 = \det \mathbf{J}_0$ and $\mathbf{C}_0 = \mathbf{J}_0^2 - \mathbf{J}_0 \operatorname{tr} \mathbf{J}_0 + \frac{1}{2}\mathbf{I}[(\operatorname{tr} \mathbf{J}_0)^2 - \operatorname{tr}(\mathbf{J}_0^2)]$ correspond to the base system properties. The first-order coefficient $c_1 = \operatorname{tr}(\mathbf{J}_0^2\mathbf{J}_1) - \frac{1}{2}[\operatorname{tr}(\mathbf{J}_0^2) - (\operatorname{tr} \mathbf{J}_0)^2] \operatorname{tr} \mathbf{J}_1 - \operatorname{tr} \mathbf{J}_0 \operatorname{tr}(\mathbf{J}_0\mathbf{J}_1)$ captures complex interactions between the base and parametric matrices, while the corresponding adjugate term $\mathbf{C}_1 = \mathbf{I}(\operatorname{tr} \mathbf{J}_0 \operatorname{tr} \mathbf{J}_1 - \operatorname{tr}(\mathbf{J}_0\mathbf{J}_1)) - (\mathbf{J}_0 \operatorname{tr} \mathbf{J}_1 + \mathbf{J}_1 \operatorname{tr} \mathbf{J}_0) + \mathbf{J}_0\mathbf{J}_1 + \mathbf{J}_1\mathbf{J}_0$ maintains the fundamental identity $\mathbf{J} \ \operatorname{adj} \mathbf{J} = \mathbf{I} \ \det \mathbf{J}$ throughout the expansion. The coefficients c_2, c_3 and $\mathbf{C}_2$ follow by symmetry. Finally, we express $(\det \mathbf{J})^{-1}$ as a power series in μ, given that $0 < \det \mathbf{J} < 2c_0$, as follows:

$$(\det \mathbf{J})^{-1} = \frac{1/c_0}{1 - [c_0 - \det \mathbf{J}]/c_0} = \frac{1}{c_0} \sum_{\kappa=0}^{\infty} \left(\frac{1}{c_0}\right)^{\kappa} (c_0 - \det \mathbf{J})^{\kappa} = \sum_{\kappa=0}^{\infty} a_\kappa \mu^\kappa \,,$$

where the polynomial coefficients are as follows

$$a_\kappa = \frac{1}{c_0} \sum_{\substack{\sigma_1,\sigma_2,\sigma_3 \in \mathbb{N}_0 \\ \sigma_1 + 2\sigma_2 + 3\sigma_3 = \kappa}} \frac{(\sigma_1+\sigma_2+\sigma_3)!}{\sigma_1!\,\sigma_2!\,\sigma_3!} \left(\frac{-c_1}{c_0}\right)^{\sigma_1} \left(\frac{-c_2}{c_0}\right)^{\sigma_2} \left(\frac{-c_3}{c_0}\right)^{\sigma_3} .$$

These mathematical formulations provide the foundation for efficient computation of system responses in parametric studies, where the polynomial expansions allow for rapid evaluation of determinant and inverse-like operations without explicit matrix inversion at each parameter value. The implementation leverages algebraic identities to minimize computational cost while maintaining numerical stability, which is crucial for large-scale finite element applications in structural dynamics where parameter studies and frequency sweeps are common analysis techniques.

Finite Element Discretisation

Closed-form solutions for elastodynamic problems are generally unavailable, necessitating numerical discretisation. The Rayleigh-Ritz method approximates the displacement field using spatially admissible functions with time-dependent coefficients:

$$\mathbf{u}(\mathbf{x}_0, t) \approx \mathbf{N}(\mathbf{x}_0)\mathbf{U}(t),$$

where $\mathbf{N}(\mathbf{x}_0)$ contains shape functions satisfying homogeneous Dirichlet conditions, and square-integrability over V. Natural boundary conditions are enforced weakly via the variational formulation.

The Finite Element Method implements this framework through domain partitioning into elements. Within each element, displacements interpolate nodal values:

$$\mathbf{u}(\mathbf{x}_0, t) \approx {}^{\mathrm{e}}\mathbf{N}(\mathbf{x}_0)\,{}^{\mathrm{e}}\mathbf{U},$$

with virtual displacements discretised similarly. Global assembly enforces kinematic compatibility through shared nodal displacements and force equilibrium through summation of element contributions, yielding global matrices $\mathbf{M}$, $\mathbf{K}$, and force vectors. For linear hyperelasticity, the internal force exhibits cubic nonlinearity:

$$\mathbf{F}^{\mathrm{int}} = \mathbf{K}\mathbf{U} + \mathbf{G}(\mathbf{U}^{\otimes 2}) + \mathbf{H}(\mathbf{U}^{\otimes 3}),$$

where $\mathbf{G}$ and $\mathbf{H}$ are symmetric tensors representing quadratic and cubic stiffness terms, and $\blacksquare^{\otimes m}$ represents the m-th Kronecker power (repeated Kronecker product with itself). The index permutation of the tensors is fully symmetric and reduces computational storage and ensures energy conservation, with force components given by:

$$\mathbf{G}(\mathbf{U}^{\otimes 2}) = \sum_{r \le s} \mathbf{G}_{\{r,s\}} U_r U_s \qquad \text{and} \qquad \mathbf{H}(\mathbf{U}^{\otimes 3}) = \sum_{r \le s \le t} \mathbf{H}_{\{r,s,t\}} U_r U_s U_t\,.$$

Now, considering the polynomial expansion of the parameter, at order μ^0, the equations of motion simplify to the standard governing equations of solid mechanics. However, at order μ^1, the linear terms, i.e. the inertia and the linear stiffness force are given by

$$\begin{aligned}&\int_{V_0} \rho\,\delta\mathbf{u}^\top \ddot{\mathbf{u}}\, c_1\,\mathrm{d}V_0 + \int_{V_0} (\nabla_0 \delta\mathbf{u}\,\mathbf{C}_1)_{\mathrm{sym}} : \mathcal{A} : (\nabla_0 \mathbf{u}\,\mathbf{C}_0)_{\mathrm{sym}}\, a_0 \\ &+ (\nabla_0 \delta\mathbf{u}\,\mathbf{C}_0)_{\mathrm{sym}} : \mathcal{A} : (\nabla_0 \mathbf{u}\,\mathbf{C}_1)_{\mathrm{sym}}\, a_0 + (\nabla_0 \delta\mathbf{u}\,\mathbf{C}_0)_{\mathrm{sym}} : \mathcal{A} : (\nabla_0 \mathbf{u}\,\mathbf{C}_0)_{\mathrm{sym}}\, a_1\,\mathrm{d}V_0\,.\end{aligned}$$

The quadratic stiffness term expands to eight terms and the cubic stiffness term to five terms, at linear order of μ. Because of the inverse of the displacement gradient, the stiffness terms expand to infinite order. However, the inertia and the external terms only expand to cubic order on μ and have only one contribution per coefficient.

Elastic Force Tensors

To derive the tensors of the elastic force, we separate the orders of the first Piola-Kirchhoff stress tensor $\mathbf{P}$. For instance, its linear part $\mathbf{P}_1$ is as follows

$$\mathbf{P}_1 = \lambda\,\mathbf{I}\,\mathrm{tr}(\nabla\mathbf{u}) + \mu\,(\nabla\mathbf{u} + \nabla\mathbf{u}^\top) = \sum_j \Big[\lambda\,\mathbf{I}\,(\nabla N_j \mathbf{U}_j) + \mu\,\big(\mathbf{U}_j \nabla N_j + \nabla N_j^\top \mathbf{U}_j^\top\big)\Big]\,.$$

The linear part of the elasticity term in the weak form integrand yields

$$\begin{aligned}&\nabla\delta\mathbf{u}^\top : \mathbf{P}_1 = \mathrm{tr}\big(\nabla\delta\mathbf{u}\,\mathbf{P}_1^\top\big) = \lambda\,\mathrm{tr}(\nabla\delta\mathbf{u})\ \mathrm{tr}(\nabla\mathbf{u}) + \mu\,\mathrm{tr}(\nabla\delta\mathbf{u}\,\nabla\mathbf{u}) + \mu\,\mathrm{tr}\big(\nabla\delta\mathbf{u}\,\nabla\mathbf{u}^\top\big) = \\ &\sum_{i,j} \Big[\lambda\,\mathrm{tr}(\delta\mathbf{U}_i \nabla N_i)\ \mathrm{tr}(\mathbf{U}_j \nabla N_j) + \mu\,\mathrm{tr}(\delta\mathbf{U}_i \nabla N_i \mathbf{U}_j \nabla N_j) + \mu\,\mathrm{tr}\big(\delta\mathbf{U}_i \nabla N_i \nabla N_j^\top \mathbf{U}_j^\top\big)\Big] = \\ &\sum_{i,j} \Big[\lambda \delta\mathbf{U}_i^\top \nabla N_i^\top \nabla N_j \mathbf{U}_j + \mu \delta\mathbf{U}_i^\top \nabla N_j^\top \nabla N_i \mathbf{U}_j + \mu\,\nabla N_i \nabla N_j^\top\,\delta\mathbf{U}_i^\top \mathbf{U}_j\Big]\end{aligned}$$

Finally, integrating over the (elemental) domain yields the stiffness matrix

$$\int_V \mathrm{tr}(\nabla\delta\mathbf{u}\,\mathbf{P}_1)\,\mathrm{d}V = \sum_{i,j} \delta\mathbf{U}_i^\top \Big(\underbrace{\lambda\boldsymbol{\mathcal{B}}_{ij}^\top + \mu\boldsymbol{\mathcal{B}}_{ij} + \mu\,\mathrm{tr}(\boldsymbol{\mathcal{B}}_{ij})\mathbf{I}}_{\mathbf{K}_{ij}}\Big)\mathbf{U}_j \qquad \text{with} \qquad \boldsymbol{\mathcal{B}}_{ij} = \int_V \nabla N_j^\top \nabla N_i\,\mathrm{d}V\,,$$

where $\boldsymbol{\mathcal{B}}_{ji} = \boldsymbol{\mathcal{B}}_{ij}^\top$ and $\mathbf{K}_{ji} = \mathbf{K}_{ij}^\top$. Expressly, the stiffness matrix felt by node i due to displacements in node j is:

$$\boxed{\mathbf{K}_{ij} = \lambda\boldsymbol{\mathcal{B}}_{ij}^\top + \mu\boldsymbol{\mathcal{B}}_{ij} + \mu\,\mathrm{tr}(\boldsymbol{\mathcal{B}}_{ij})\mathbf{I}}\,.$$

For the tensor of the quadratic term we have

$$\mathbf{P}_2 = \lambda\,\nabla\mathbf{u}\,\mathrm{tr}(\nabla\mathbf{u}) + \mu\,\nabla\mathbf{u}\,(\nabla\mathbf{u}^\top + \nabla\mathbf{u}) + \lambda\,\mathbf{I}\,\mathrm{tr}\big(\nabla\mathbf{u}^\top\,\nabla\mathbf{u}\big)/2 + \mu\,\nabla\mathbf{u}^\top\,\nabla\mathbf{u} =$$
$$\sum_{j,k}\Big[\lambda\,\mathbf{U}_j\nabla N_j\,\mathrm{tr}(\mathbf{U}_k\nabla N_k) + \mu\,\mathbf{U}_j\nabla N_j\,(\nabla N_k^\top\mathbf{U}_k^\top + \mathbf{U}_k\nabla N_k) + \lambda\,\mathbf{I}\,\mathrm{tr}\big(\nabla N_j^\top\mathbf{U}_j^\top\mathbf{U}_k\nabla N_k\big)/2 + \mu\,\nabla N_j^\top\mathbf{U}_j^\top\mathbf{U}_k\nabla N_k\Big]\,.$$

And the second order term in the weak form integrand becomes

$$\begin{aligned}\mathrm{tr}(\nabla\delta\mathbf{u}\,\mathbf{P}_2) = \lambda\;\mathrm{tr}(\nabla\delta\mathbf{u}\,\nabla\mathbf{u})\,\mathrm{tr}(\nabla\mathbf{u}) + \mu\;\mathrm{tr}\big(\nabla\delta\mathbf{u}\,\nabla\mathbf{u}\,\nabla\mathbf{u}^\top\big) + \mu\;\mathrm{tr}(\nabla\delta\mathbf{u}\,\nabla\mathbf{u}\,\nabla\mathbf{u})\\ +\lambda\;\mathrm{tr}(\nabla\delta\mathbf{u})\,\mathrm{tr}\big(\nabla\mathbf{u}^\top\,\nabla\mathbf{u}\big)/2 + \mu\;\mathrm{tr}\big(\nabla\delta\mathbf{u}\,\nabla\mathbf{u}^\top\,\nabla\mathbf{u}\big)\end{aligned}$$

Hence, the tensor of the quadratic term writes

$$\mathbf{G}_{ijk} = \lambda\,\boldsymbol{\mathcal{B}}_{ijk} + \mu\Big[\mathop{\mathrm{tr}}_{jk}\boldsymbol{\mathcal{B}}_{ijk}\otimes\mathbf{I}\Big]^\top + \mu\,\boldsymbol{\mathcal{B}}_{ijk}^\top + \frac{\lambda}{2}\Big[\mathop{\mathrm{tr}}_{jk}\boldsymbol{\mathcal{B}}_{ijk}\otimes\mathbf{I}\Big] + \mu\Big[\mathop{\mathrm{tr}}_{ij}\boldsymbol{\mathcal{B}}_{ijk}\otimes\mathbf{I}\Big]$$

$$\text{with}\qquad \boldsymbol{\mathcal{B}}_{ijk} = \int_V \nabla N_j\otimes\nabla N_i\otimes\nabla N_k\,\mathrm{d}V \qquad\text{such that}\qquad \int_V \mathrm{tr}(\nabla\delta\mathbf{u}\,\mathbf{P}_2)\,\mathrm{d}V = \sum_{i,j,k}\mathbf{G}_{ijk}(\delta\mathbf{U}_i\otimes\mathbf{U}_j\otimes\mathbf{U}_k)\,.$$

Here, $\mathop{\mathrm{tr}}_{jk}$ denotes the trace w.r.t. the second and third indices, j and k.

A similar development can be done for the cubic term, such that $\mathbf{H}_{ijkl}$—the tensor that describes the force felt in node i due to the cubic contribution of the combined displacements of nodes j, k and l—can be constructed from a linear transformation of the following tensor:

$$\boldsymbol{\mathcal{B}}_{ijkl} = \int_V \nabla N_j\otimes\nabla N_i\otimes\nabla N_k\otimes\nabla N_l\,\mathrm{d}V\,.$$

Recursive Computation of Parametric Elastic Force Tensors

We integrate the tensor product of the shape functions to build the $\boldsymbol{\mathcal{B}}$ tensors, which can be efficiently transformed into the polynomial tensors of the elastic force for assembly. For the parametric study, these integrals are reformulated in the reference domain using the identities $\mathrm{d}V = \det\mathbf{J}\;\mathrm{d}V_0$ and $\nabla\blacksquare = \nabla_0\blacksquare\cdot\mathrm{adj}\,\mathbf{J}\;(\det\mathbf{J})^{-1}$, as

$$\boldsymbol{\mathcal{B}}_{i_1\ldots i_\gamma} = \int_V \nabla N_{i_1}(\mathbf{x})\otimes\cdots\otimes\nabla N_{i_\gamma}(\mathbf{x})\mathrm{d}V = \int_{V_0}\big[\nabla_0 N_{i_1}(\mathbf{x}_0)\;\mathrm{adj}\,\mathbf{J}(\mathbf{x}_0)\big]\otimes\cdots\otimes\big[\nabla N_{i_\gamma}(\mathbf{x}_0)\;\mathrm{adj}\,\mathbf{J}(\mathbf{x}_0)\big](\det\mathbf{J}(\mathbf{x}_0))^{-\gamma}\mathrm{d}V_0$$

To optimize numerical quadrature, we separate from the integrand the tensor product of shape functions for the force term of order γ, which marks the force in node i_1 due to the combined displacements of nodes i_2, ... and $i_{\gamma+1}$:

$$\boldsymbol{\mathcal{N}}_{i_1,\ldots,i_{\gamma+1}} = \big[\nabla_0 N_{i_1}\;\mathrm{adj}\,\mathbf{J}\big]\otimes\cdots\otimes\big[\nabla N_{i_\gamma}\;\mathrm{adj}\,\mathbf{J}\big] = \boldsymbol{\mathcal{N}}_{i_1,\ldots,i_\gamma}\otimes\big[\nabla_0 N_{i_\gamma}\;\mathrm{adj}\,\mathbf{J}\big]\,.$$

In the context of FEM for structural dynamics, particularly when dealing with nonlinear problems and parametric studies, we employ a recursive algorithm for constructing higher-order force tensors. This approach systematically constructs force contributions through a hierarchical tensor product operation, enabling efficient computation of perturbation expansions or homotopy analysis solutions. The mathematical foundation of this algorithm rests on a recursive convolution structure that builds higher-order terms from lower-order components. Let $\boldsymbol{\mathcal{N}}^{(q)}_{i_1,\ldots i_{\gamma+1}}$ represent the tensor product of shape functions for the force term of order γ and polynomial degree q with respect to the geometry parameter μ. The core operation is a discrete convolution representing tensor contraction between successive force orders. The recursive relation is defined as:

$$\boldsymbol{\mathcal{N}}^{(q)}_{i_1,\ldots,i_{\gamma+1}} = \sum_{\ell=\max\{0,q-\Gamma_1\}}^{\min\{\Gamma_\gamma,q\}}\boldsymbol{\mathcal{N}}^{(\ell)}_{i_1,\ldots,i_\gamma}\otimes\big[\nabla_0 N_{i_{\gamma+1}}\;\mathrm{adj}\,\mathbf{J}|_{q-\ell}\big]$$

where $\mathrm{adj}\,\mathbf{J}|_{q-\ell}$ denotes the coefficient of degree $q-\ell$ in μ. We build these tensors for degrees $q = 0,\ldots,\Gamma_\gamma$, where the maximum permissible degree at each force order is determined by $\Gamma_\gamma = \min\big\{\gamma\cdot(d-1)\cdot\deg_\mu\mathbf{J},\,\Gamma_{\max}\big\}$, where d represents the spatial dimension (e.g. 3), $\Gamma_{\max}$ is the maximum parametric expansion order in the procedure, and $\deg_\mu\mathbf{J}(\mu)$

is 1 in the examples above. This formulation creates a triangular degree structure that prevents combinatorial explosion while maintaining approximation accuracy.

The algorithm implements a depth-first traversal of the force order tree, beginning with initial conditions $\boldsymbol{\mathcal{N}}_{i_1}$ and progressively building higher-order terms. At each recursion level γ, the algorithm computes the tensor product between the previous order's result and available base operators, then proceeds to the next force order. This continues until reaching the maximum specified force order, at which point the leaf nodes are processed and the final contributions are assembled.

The number of float multiplications (time complexity) and memory required (space complexity) for computing $\boldsymbol{\mathcal{N}}^{(q)}_{i_1,\ldots,i_{\gamma+1}}$ are both $\mathcal{O}(d^{\gamma+1}\, S_{\gamma,q})$, where $S_{\gamma,q}$ is the number of terms in the summation: $q+1$ if $q < \Gamma_1$, $\Gamma_1 + 1$ if $\Gamma_1 \leq q \leq \Gamma_\gamma$, and $\Gamma_1 + \Gamma_\gamma - q + 1$ otherwise. The naive approach, that does not reuse lower order data, scales like $d^{2(\gamma+1)} \binom{q+\gamma}{\gamma}$. We compute $\boldsymbol{\mathcal{N}}^{(q)}_{i_1,\ldots,i_{\gamma+1}}$ for all multisets (unordered combinations) with $\gamma+1$ nodes, which increases the cost by a factor of $\binom{N_{\text{Nodes}}+\gamma}{\gamma+1}$.

This recursive framework is particularly valuable in structural dynamics for handling geometric nonlinearities, material nonlinearities, and parametric uncertainties. The method provides a systematic approach to computing higher-order force terms in perturbation-based solutions, asymptotic expansions, or homotopy analysis methods applied to nonlinear finite element formulations. The algorithm's structure naturally accommodates adaptive refinement strategies and can be parallelized across different force orders for improved computational performance in large-scale structural dynamics simulations.

For numerical quadrature, we loop over the elemental integration points and for each, we precompute $\boldsymbol{\mathcal{N}}_i = \nabla_0 N_i \ \mathrm{adj}\, \mathbf{J}$ for all the nodes and for all relevant parametric degrees; then for each node in the element we start the recursion. In this manner, it is possible to build the elemental tensors with minimal computation. Naturally, memory limitations prohibit assembling these tensors in domains with many elements. Nonetheless, the framework remains as a useful tool for optimization in parametric FEM and specifically for polynomial based parametrisation methods for ROM.

Conclusion

This work introduces a framework for the parametric reduced-order modelling of finite element methods, specifically addressing geometric variations controlled by a single parameter. The formulation ensures that geometric changes do not alter intrinsic material properties, stress states, or introduce spurious forces. The methodology involves a power series expansion of the inverse determinant from the weak form, with explicit derivations for the zeroth and first-order terms. A recursive computational strategy is presented for the efficient assembly of the elemental polynomial tensors representing the nonlinear elastic force. Additionally, external forcing and parametric dependence are incorporated into an enlarged autonomous system, enabling reduction through the direct parametrisation of invariant manifolds, which involves solving a sequence of homological equations.

Disclosure statement:
No potential conflict of interest is reported by the authors.

References

[1] Xavier Cabré, Ernest Fontich, and Rafael De la Llave. "The parameterization method for invariant manifolds I: manifolds associated to non-resonant subspaces". In: *Indiana University mathematics journal* (2003), pp. 283–328.

[2] Xavier Cabré, Ernest Fontich, and Rafael De la Llave. "The parameterization method for invariant manifolds II: regularity with respect to parameters". In: *Indiana University mathematics journal* (2003), pp. 329–360.

[3] Xavier Cabré, Ernest Fontich, and Rafael De La Llave. "The parameterization method for invariant manifolds III: overview and applications". In: *Journal of Differential Equations* 218.2 (2005), pp. 444–515.

[4] Alex Haro, Marta Canadell, Jordi-Lluis Figueras, Alejandro Luque, and Josep-Maria Mondelo. "The parameterization method for invariant manifolds". In: *Applied mathematical sciences* 195 (2016).

[5] John Guckenheimer and Philip Holmes. *Nonlinear oscillations, dynamical systems, and bifurcations of vector fields*. Vol. 42. Springer Science & Business Media, 2013.

[6] Cyril Touzé, Olivier Thomas, and Antoine Chaigne. "Hardening/softening behaviour in non-linear oscillations of structural systems using non-linear normal modes". In: *Journal of Sound and Vibration* 273.1-2 (2004), pp. 77–101.

[7] George Haller and Sten Ponsioen. "Nonlinear normal modes and spectral submanifolds: existence, uniqueness and use in model reduction". In: *Nonlinear dynamics* 86 (2016), pp. 1493–1534.

[8] Andrea Opreni, Alessandra Vizzaccaro, Cyril Touzé, and Attilio Frangi. "High-order direct parametrisation of invariant manifolds for model order reduction of finite element structures: application to generic forcing terms and parametrically excited systems". In: *Nonlinear Dynamics* 111.6 (2023), pp. 5401–5447.

[9] Aurélien Grolet, Alessandra Vizzaccaro, Marielle Debeurre, and Olivier Thomas. "High order invariant manifold model reduction for systems with non-polynomial non-linearities: geometrically exact finite element structures and validity limit". In: (2024).

[10] Leonardo Bettini, Mattia Cenedese, and George Haller. "Model reduction to spectral submanifolds in piecewise smooth dynamical systems". In: *International Journal of Non-Linear Mechanics* 163 (2024), p. 104753.

[11] Alessandra Vizzaccaro, Giorgio Gobat, Attilio Frangi, and Cyril Touzé. "Direct parametrisation of invariant manifolds for non-autonomous forced systems including superharmonic resonances". In: *Nonlinear Dynamics* 112.8 (2024), pp. 6255–6290.

[12] Mingwu Li, Thomas Thurnher, Zhenwei Xu, and Shobhit Jain. "Data-free non-intrusive model reduction for nonlinear finite element models via spectral submanifolds". In: *Computer Methods in Applied Mechanics and Engineering* 434 (2025), p. 117590.

[13] Jacopo Marconi, Paolo Tiso, Davide E Quadrelli, and Francesco Braghin. "A higher-order parametric nonlinear reduced-order model for imperfect structures using Neumann expansion". In: *Nonlinear Dynamics* 104.4 (2021), pp. 3039–3063.

[14] Ahmed Amr Morsy and Paolo Tiso. "Predicting the variability of the dynamics of bolted joints using polynomial chaos expansion". In: *Mechanical Systems and Signal Processing* 224 (2025), p. 112008.

[15] R. Anderson, J. Andrej, A. Barker, J. Bramwell, J.-S. Camier, J. Cerveny, V. Dobrev, Y. Dudouit, A. Fisher, Tz. Kolev, W. Pazner, M. Stowell, V. Tomov, I. Akkerman, J. Dahm, D. Medina, and S. Zampini. "MFEM: A Modular Finite Element Methods Library". In: *Computers & Mathematics with Applications* 81 (2021), pp. 42–74.

[16] Attilio Frangi, Alessio Colombo, Alessandra Vizzaccaro, and Cyril Touzé. "Reduced Order Modelling of Fully Coupled Electro-Mechanical Systems Through Invariant Manifolds With Applications to Microstructures". In: *International Journal for Numerical Methods in Engineering* 126.3 (2025), e7641.