Chapter 4
Machine Learning for Experimental Bifurcation Analysis

Grégoire Bourdouch, Pierre Geurts, and Gaëtan Kerschen

Abstract Nonlinear mechanical structures can exhibit multiple coexisting periodic solutions, including unstable responses that remain inaccessible to standard experimental testing. Conventional swept-sine measurements, rooted in linear assumptions, capture only the stable portions of the frequency response and thus overlook essential bifurcations and unstable dynamics. Beyond the fundamental resonance, nonlinear systems also resonate at integer multiples or fractions of the excitation frequency, giving rise to superharmonic and subharmonic responses, some of which may appear as isolated branches.

This work proposes a data-driven framework that leverages deep convolutional networks to complete the missing information from open-loop measurements. From open-loop experimental data alone, the method infers the full nonlinear frequency response. Experimental validation on an electronic Duffing oscillator confirms the accuracy of the predictions, while the framework can be readily extended by incorporating additional nonlinearities into the training dataset, paving the way toward a generic tool for nonlinear experimental analysis.

Keywords Nonlinear vibration testing · Secondary resonances · Isola · Machine learning · Convolutional neural networks

Introduction

Aerospace and other engineering structures may exhibit nonlinear dynamical behaviors. Geometric nonlinearities, friction, or material properties give rise to multi-stability, bifurcations, and sudden jumps between responses. Linear models and assumptions, which form the basis of conventional modal analysis, therefore fall short when applied to these systems.

Experimental data remain essential for identifying nonlinear responses, yet standard open-loop methods such as stepped-sine and swept-sine testing can only access the stable portions of the nonlinear frequency response curves (FRCs). Unstable and isolated branches, however, carry critical information about the dynamics but remain experimentally inaccessible in open-loop.

To address this limitation, several control-based approaches have been developed. Phase-locked loop (PLL) [1] testing, control-based continuation (CBC) [2], and real-time control (RTC) [3] methods use feedback control to stabilize otherwise unstable responses and thereby enable the construction of complete FRCs. While effective in principle, these methods face practical challenges such as controller instabilities, sensitivity to noise, and the need for careful gain tuning, which limit their robustness and ease of use in experimental practice.

These limitations motivate the exploration of alternative strategies. In this work, we propose a data-driven framework that uses deep learning to complete the missing information from open-loop measurements. From standard experimental data alone, the method can infer the unstable and isolated parts of the nonlinear frequency response, offering a robust and simple alternative to traditional control-based approaches.

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Background and Methodology

This work employs a supervised learning framework in which the model parameters $\hat{\phi}$ are identified by minimizing a prescribed loss function, with the ultimate objective of achieving generalization, i.e., ensuring accurate predictions on previously unseen examples. A central innovation lies in reformulating FRCs as abstracted image-based representations rather than direct plots of measured amplitudes. This enables the problem to be cast as an image-to-image translation task, where experimentally accessible responses obtained from open-loop tests are mapped to the corresponding nonlinear features that remain hidden. Convolutional neural networks are particularly suited for this setting, as they efficiently exploit local spatial correlations in high-dimensional data, in contrast to fully connected networks that become computationally prohibitive. Among the available architectures, U-Net [4] has proven especially effective. It adopts a symmetric encoder–decoder structure: the encoder compresses the input into progressively more abstract features, while the decoder reconstructs spatial resolution, with skip connections preserving fine details lost during encoding. This combination allows for accurate mapping of input to output in image translation problems. A schematic representation of the deep learning framework employed in this work is shown in Fig. 1.

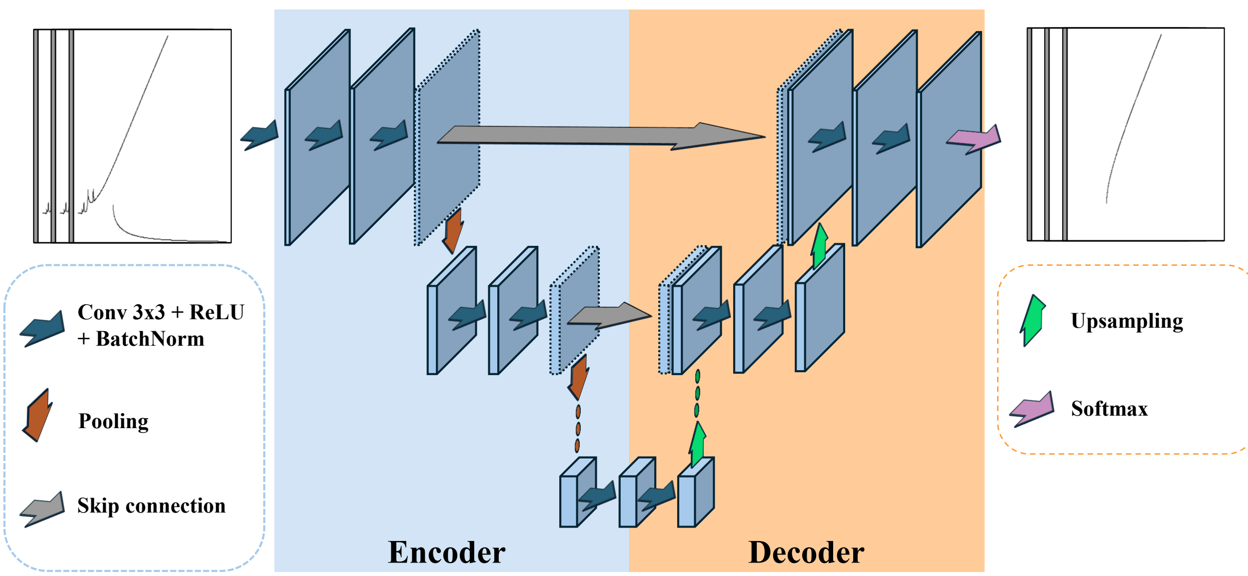

Fig. 1 Schematic of the U-Net framework for predicting the unstable branch of the fundamental resonance

The mandatory training process of a deep learning model requires a reliable dataset, yet no standard databases of FRCs exist. In this work, FRCs are generated numerically using the harmonic balance method with pseudo-arclength continuation [5], which captures both stable and unstable branches by tracking solutions through fold points. Additional disconnected solutions, such as isolated subharmonic responses, are obtained with the two-harmonic homotopy method [6]. The database is restricted to single-degree-of-freedom (SDOF) oscillators, ensuring immediate experimental applicability to SDOF setups as well as to multi-degree-of-freedom (MDOF) systems in which modal interactions remain absent. To guarantee physical relevance, the dataset incorporates canonical nonlinear oscillators: the Duffing oscillator to model cubic hardening nonlinearities in clamped–clamped beams, the Helmholtz–Duffing oscillator to introduce quadratic contributions, and piecewise linear stiffness models to represent non-smooth restoring forces such as those arising in contact problems.

Together, these numerical schemes provide a comprehensive collection of FRCs from which stable and unstable branches are extracted. The Bresenham line algorithm is then employed to discretize these curves into binary images of resolution 320×320, chosen for this study but adjustable to higher resolutions if required. Separate models are trained for the different dynamical features of interest, namely the unstable branches of the fundamental and superharmonic responses, and the isolated subharmonic responses. To support this framework, dedicated window encodings were designed [7], each relying exclusively on data obtainable from open-loop tests, thereby ensuring experimental applicability. Finally, the predictions of the individual models are combined to reconstruct the complete FRC.

Analysis

The method was experimentally validated on an analog electronic circuit replicating the Duffing oscillator behavior, which is subjected to harmonic forcing. The setup, shown in Fig. 2, was designed to exhibit minimal dissipation, pronounced nonlinearities, and relatively low resonance frequencies. This setup also offers a significant advantage, as it is not subject to shaker–structure interactions. This circuit is governed by an equation equivalent to the Duffing oscillator, written as

$$m\,\ddot{x} + c\,\dot{x} + k\,x + k_3\,x^3 \;=\; f\,\sin(\omega t) + f_0, \tag{1}$$

where the effective parameters of the equivalent oscillator are $m = 10^{-4}$ [s^2], $c = 4.9 \times 10^{-4}$ [s], $k = 1.68$ [$-$], $k_3 = 0.983$ [V^{-2}], and the forcing amplitude is $f = 2$ [V]. Here, f_0 denotes a static input voltage.

The presence of f_0 induces a static deflection, which breaks the odd symmetry of the pure Duffing nonlinearity and introduces an effective quadratic stiffness term [8]. This symmetry breaking modifies the bifurcation structure and enables even-order nonlinear phenomena, such as the 2:1 superharmonic resonance, which are absent in the symmetric Duffing case and thus not represented in the training set.

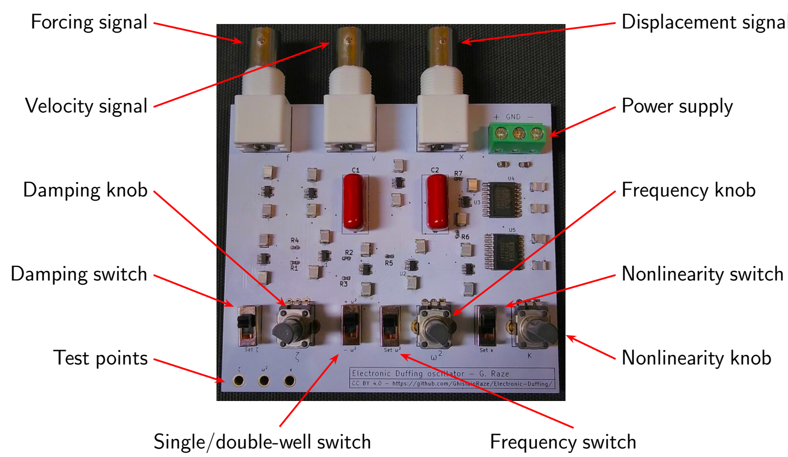

Fig. 2 Schematic of the electronic Duffing oscillator used in this study, adapted from [9]

A key advantage of the proposed method lies in its straightforward experimental implementation. Unlike control-based techniques, which often require extensive parameter tuning, the present framework relies solely on open-loop tests. In this work, swept-sine experiments were employed, a standard feature in most commercial vibration software. From these measurements, the response envelope is extracted to identify the stable branches of the frequency response, as illustrated in Fig. 3, where the swept-up and swept-down responses at an excitation amplitude of $f = 2$ [V] are displayed, and the stable branches are highlighted in red. The swept-up and -down sine tests capture the characteristic jump phenomena of the system, corresponding respectively to the downward and upward transitions between coexisting solutions. In addition, superharmonic resonances are observed at frequencies greater than one-third and one-half of the natural frequency, reflecting the strong nonlinear behavior of the oscillator.

Inference is performed on input images representing the stable branches, constructed from swept-sine envelopes in the same manner as for the numerically generated training samples. For validation, the predictions are compared against results obtained with the (x-)ACBC algorithm [10] on the same experimental setup. Fig. 4(a) shows the 3:1 superharmonic resonance, Fig. 4(b) the 2:1 superharmonic resonance, and Fig. 4(c) the fundamental resonance (1:1) together with the isolated 1:3 subharmonic resonance.

The results highlight the capability of the proposed framework to reconstruct nonlinear resonances with high accuracy. For the fundamental resonance, the predictions are in excellent agreement with the experimental measurements, while for the 3:1 superharmonic the correspondence is particularly strong even in the vicinity of the folds, where continuation techniques typically encounter difficulties. The 2:1 superharmonic resonance is less accurately captured, a discrepancy attributable to differences between the experimental configuration and the numerical systems employed during training. Most notably, the framework successfully predicts the complete isolated branch associated with the 1:3 subharmonic resonance. This constitutes a significant result, as subharmonic responses are notoriously difficult to capture experimentally: unlike fundamental

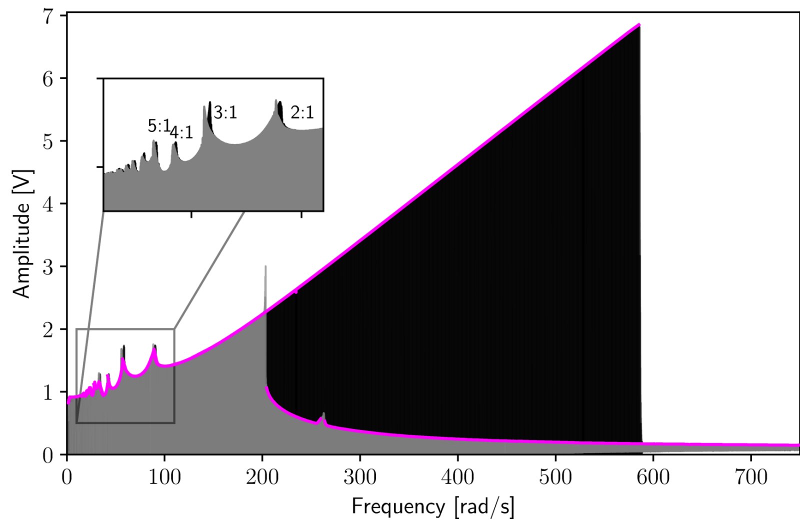

Fig. 3 Swept-up (▬), swept-down (▬) sine test responses of the electronic Duffing oscillator at $f = 2$ [V], with the stable branches highlighted (▬)

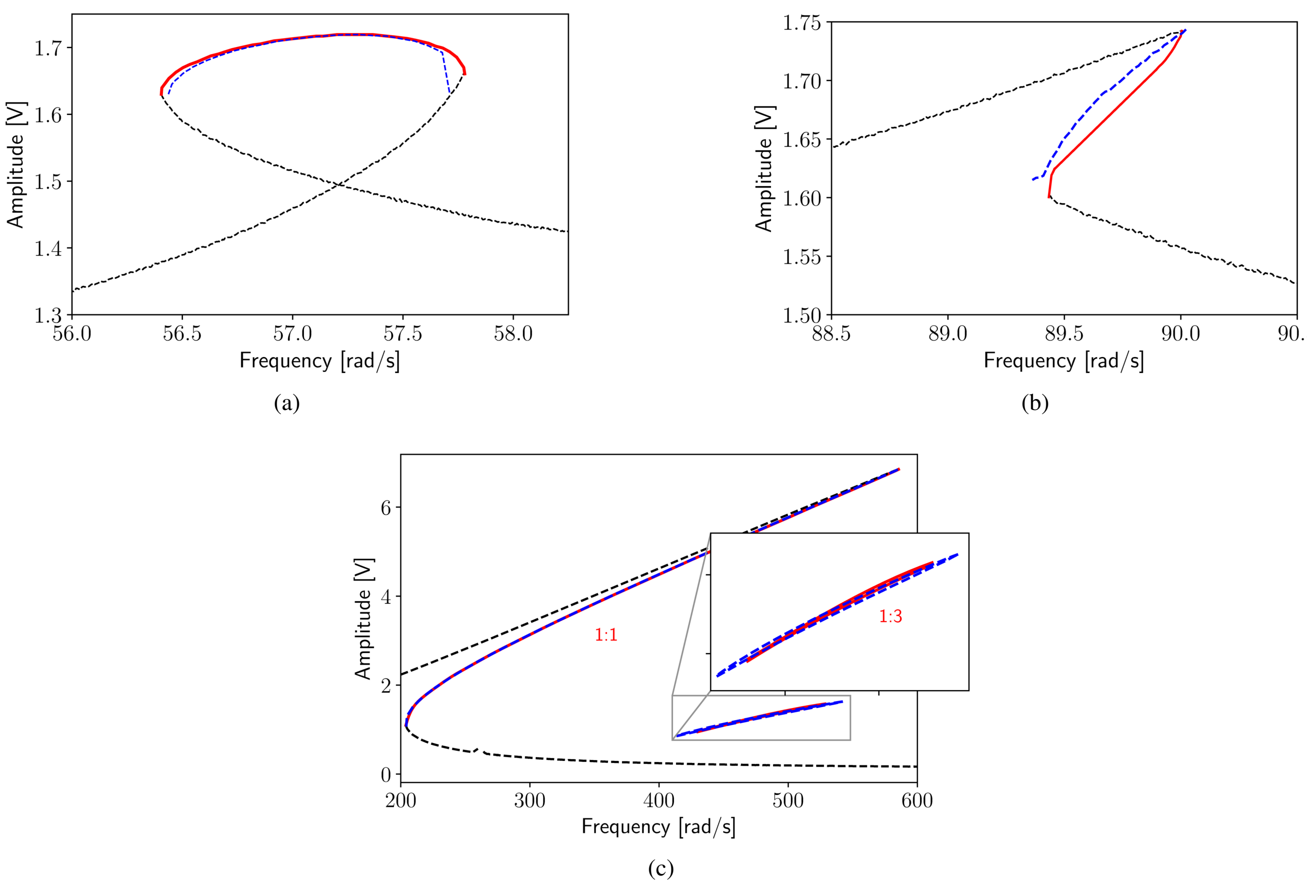

Fig. 4 FRC predictions (▬), measurements with x-ACBC (▬), and stable branches (▬ ▬) of the electronic Duffing oscillator at $f = 2$ [V]. (a) Superharmonic 3:1, (b) superharmonic 2:1, and (c) fundamental 1:1 with subharmonic 1:3

or superharmonic resonances, they do not lie on the direct continuation path of the primary frequency response and are consequently challenging to identify using continuation methods. By combining the model predictions with the experimentally measured stable branches, the complete nonlinear frequency response curve can be reconstructed, as illustrated in Fig. 5.

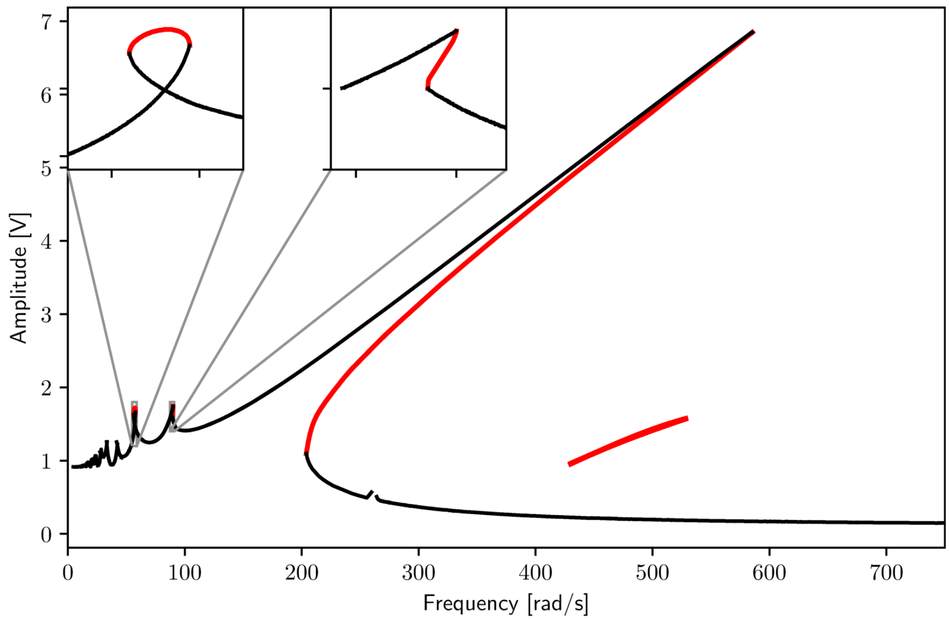

Fig. 5 Reconstructed FRC of the electronic Duffing oscillator at $f = 2$ [V]. Model predictions (red line) include fundamental, subharmonic, and superharmonic components, while swept-sine envelopes are shown as black curves (black line)

Conclusion

This work introduced a data-driven framework for nonlinear vibration testing that exploits open-loop measurements and deep learning to recover inaccessible responses. By reformulating frequency response curves as images, the problem was cast as an image-to-image translation task. The method was experimentally validated on an electronic Duffing oscillator, where it successfully reconstructed the complete frequency response, including unstable branches, superharmonic resonances, and the isolated 1:3 subharmonic resonance, a feature notoriously difficult to detect with continuation methods. These results demonstrate that the proposed approach bridges the gap between standard experimental modal analysis and the full characterization of nonlinear systems.

Future work will focus on extending the training database to encompass a broader range of nonlinear oscillators and MDOF systems, thereby enabling greater generalization across experimental configurations. This strategy paves the way toward a generic tool for nonlinear experimental analysis that complements control-based methods.

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