# Chapter 1
# Determining the Optimal Length for Impulse Response Functions Calculated through Time Domain Deconvolution

**Oliver M. Zobel and Daniel J. Rixen**

**Abstract** Using Time Domain Deconvolution, the Impulse Response Functions (IRFs) are estimated from measurements using a least-squares solution. Even when only a small part of the estimated IRFs is used for Impulse-Based Substructuring (IBS), calculating a longer IRF, i.e., using more data from the measurement, is beneficial; however, more data means higher computational costs, which for Time Domain Deconvolution increase non-linearly. This paper investigates the influence of the IRF length on IBS results using experimental data. An optimal IRF length, i.e., without unnecessary computational costs, is determined using time domain signal comparison metrics as well as adjusted variants of the Akaike and Bayesian Information Criterion. The results show that the IRF length can be significantly decreased by accepting a negligible error.

**Keywords** Impulse response functions · Time domain deconvolution · Impulse-based substructuring · Time domain substructuring · Experimental techniques

## Introduction

Having accurate estimates of a mechanical system's impulse response functions (IRFs) is crucial for successfully applying Impulse-Based Substructuring (IBS), the time domain counterpart of the established Frequency-Based Substructuring (FBS) method. To estimate IRFs, a procedure called Time Domain Deconvolution is utilized, which is the inverse of the Duhamel (convolution) integral [1]. Based on the construction of the deconvolution integral, the length of the time series of the applied force $\bar{\boldsymbol{f}}[k]$, measured response $\bar{\boldsymbol{y}}[k]$, and IRF $\bar{\boldsymbol{h}}[k]$ to be calculated are not independent. Here, the length of the IRF $t_{\mathrm{IRF}}$ is chosen as the variable and the length of the response $\bar{\boldsymbol{y}}[k]$, and force $\bar{\boldsymbol{f}}[k]$ is adjusted accordingly.

An example for the influence of the estimated IRF's length $t_{\mathrm{IRF}}$ on the result of an acceleration response prediction with IBS can be seen in fig. 1. Here, responses to an imperfect impulse are simulated using Newmark time integration of a linear FEM model, artificial noise is added, and then the IRFs are estimated using the same method as later used for the experiments, Time Domain Deconvolution, see [2] for further details on the simulation procedure. As can be seen, when an IRF with length $t_{\mathrm{IRF}} = 2.1\,\mathrm{ms}$, i.e., slightly longer than the evaluation time of IBS $t_{\mathrm{IBS}} = 2\,\mathrm{ms}$, is used in the deconvolution, the IBS prediction of the acceleration response $a_1^{(0)}[k]$ shows signs of instability starting around $1\,\mathrm{ms}$. This instability is due to the added noise; without added noise, the response predicted by IBS matches the Newmark integration of the reference system $S_0$, since then the pseudo-inverse would give the exact solution for the IRFs.

When more data is used for the IRF, for instance, with $t_{\mathrm{IRF}} = 5\,\mathrm{ms}$, the instability is reduced compared to $t_{\mathrm{IRF}} = 2.1\,\mathrm{ms}$. Even better results are found for $t_{\mathrm{IRF}} = 50\,\mathrm{ms}$ and $t_{\mathrm{IRF}} = 200\,\mathrm{ms}$. While using $t_{\mathrm{IRF}} = 50\,\mathrm{ms}$ gave significant improvements, a further increase to $t_{\mathrm{IRF}} = 200\,\mathrm{ms}$ only slightly changed the results.

The arising question is how long the estimated IRF should be to get the best estimate. Here, the IRFs should later be used for IBS to determine the response of the assembled system for a given time $t_{\mathrm{IBS}}$. To avoid truncation effects as observed in [3], the IRF length $t_{\mathrm{IRF}}$ is always chosen longer than the IBS evaluation time $t_{\mathrm{IBS}}$. While increasing the IRF length $t_{\mathrm{IRF}}$ can improve the IRF quality, since more time data is used, the computation costs also increase. Due to the required evaluation of a pseudo-inverse, the costs increase non-linearly, e.g., using a QR-factorization with Gram-Schmidt, proportional to $2mn^2$

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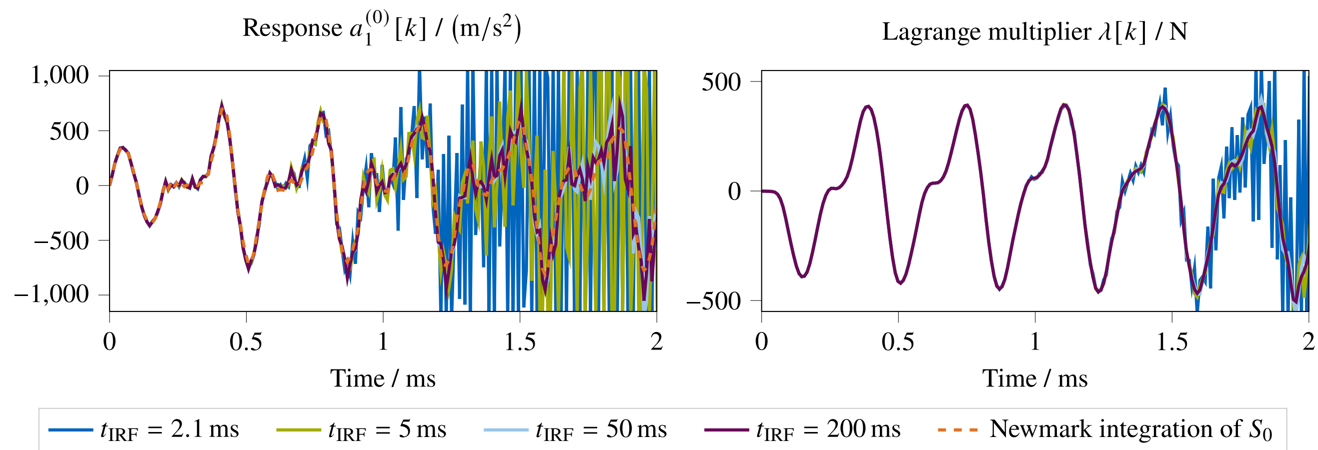


**Fig. 1** Comparison of IRFs computed from a deconvolution of responses generated by a numerical model with added noise and with variable length $t_{\mathrm{IRF}}$

(matrix $(m \times n)$) [4]. Therefore, an indicator for a good compromise between accurate IRFs and reasonable computation time should be found. This contribution investigates the effect of the IRF estimation length $t_{\mathrm{IRF}}$ on the results of IBS and tests different methods for determining the optimal length $t_{\mathrm{IRF,opt}}$ on an experimental example.

## Theory

The following section provides a brief explanation of the Impulse-Based Substructuring method, followed by an overview of the technique used for estimating the time-domain Impulse Response Function. Then, the methods used to compare signals in the time domain are outlined. Finally, procedures for determining the optimal IRF length $t_{\mathrm{IRF,opt}}$ are discussed.

### *Impulse-Based Substructuring*

The time-continuous version of the IBS method using Lagrange multipliers $\boldsymbol{\lambda}$ was given by Rixen [5] as:

**Definition Impulse-Based Substructuring Method** [5]

$$\begin{cases} \boldsymbol{y}^{(s)}(t) = \int_0^t \boldsymbol{H}^{(s)}(t-\tau)\ \left(\boldsymbol{f}^{(s)}(\tau) + \boldsymbol{B}^{(s)^{\mathrm{T}}}\boldsymbol{\lambda}(\tau)\right) \mathrm{d}\tau & \text{Equations of Motion} \\ \sum_{s=1}^{N_S} \boldsymbol{B}^{(s)}\boldsymbol{y}^{(s)}(t) = \boldsymbol{0} & \text{Compatibility} \end{cases} \tag{1}$$

where:

| | | | | | |
|---|---|---|---|---|---|
| $\boldsymbol{y}(t)$ | System response | $\boldsymbol{H}(t)$ | Matrix of impulse response functions | $(s)$ | Substructure index |
| $\boldsymbol{f}(t)$ | Externally applied forces | $\boldsymbol{B}$ | Signed Boolean constraint matrix | $*_f$ | Force mapping |
| $N_S$ | Number of substructures | $\boldsymbol{\lambda}(t)$ | Global vector of Lagrange multipliers | $*_y$ | Response mapping |

Details of the required discretization of eq. (1) can be found in [1]. The IBS scheme is evaluated using a time-stepping scheme, i.e., the convolution integral must be evaluated for every time step $k$. The calculations during one discrete time step of the IBS scheme are summarized in fig. 2. Based on the predicted response $\tilde{\boldsymbol{y}}^{(s)}[k]$, that misses the Lagrange multipliers $\boldsymbol{\lambda}[k-1]$ responsible for enforcing interface compatibility at step $k$, the current interface gap $\boldsymbol{g}[k]$ is calculated. Then, the Lagrange multipliers $\boldsymbol{\lambda}[k-1]$ that exactly close all interface gaps are calculated and applied in the corrector step [1, 5].

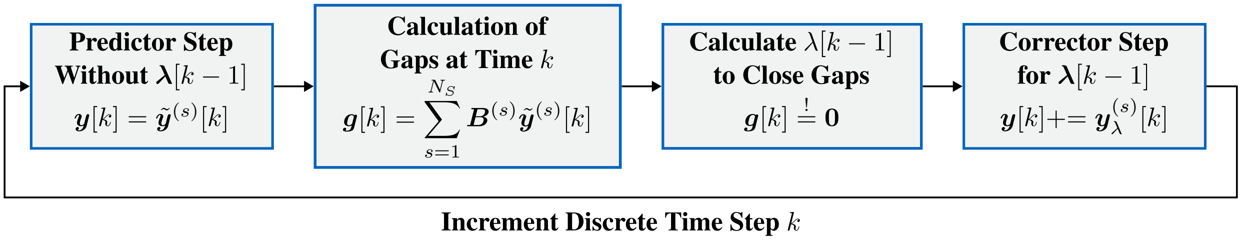


**Fig. 2** Flowchart of IBS scheme calculations within each time step

### *Time Domain Impulse Response Function Estimation*

The procedure for Time Domain Deconvolution, i.e., the IRF estimation method, is found by writing Duhamel's integral, discretized using the Cauchy product, in matrix-vector form [1, 6]:

$$\underbrace{\begin{bmatrix} y[1] \\ y[2] \\ \vdots \\ y[M] \\ \vdots \\ y[N-1] \\ y[N] \end{bmatrix}}_{\substack{\bar{\boldsymbol{y}} \\ (N\times 1)}} = \underbrace{\begin{bmatrix} f[0] & 0 & \dots & 0 \\ f[1] & f[0] & \ddots & \vdots \\ \vdots & f[1] & \ddots & 0 \\ f[M-1] & \vdots & \ddots & f[0] \\ 0 & f[M-1] & \vdots & f[1] \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & f[M-1] \end{bmatrix}}_{\substack{\boldsymbol{F} \\ (N\times Q)}} \underbrace{\begin{bmatrix} h[0] \\ h[1] \\ \vdots \\ h[Q-1] \end{bmatrix}}_{\substack{\bar{\boldsymbol{h}} \\ (Q\times 1)}} \Delta t. \tag{2}$$

Here, $\bar{\boldsymbol{y}}$ is the system response given by the convolution of the IRF $\bar{\boldsymbol{h}}$ with the force $\bar{\boldsymbol{f}}$, written as a force convolution matrix $\boldsymbol{F}$. Bar accents denote that the respective quantity is the time series of a single degree of freedom (dof) and not a physical vector, i.e., eq. (2) is for one dof. $M$ is the number of samples in the force time series greater than a noise threshold; the remaining samples are set to zero by applying a force window to reduce the influence of noise. The length of the measured signals is given by $N$, and by construction of eq. (2) from a convolution product, the length $Q$ of the IRF $\bar{\boldsymbol{h}}$ follows as $Q = N - M + 1$ [1]. In the following, the number of samples $Q$ in the estimated IRF $\bar{\boldsymbol{h}}$ is denoted with $n_{\text{IRF}}$.

A least-squares solution for the IRF $\bar{\boldsymbol{h}}$ can be found using a pseudo inverse $\boldsymbol{F}^{+}$ of the force convolution matrix $\boldsymbol{F}$:

$$\bar{\boldsymbol{h}} = \frac{1}{\Delta t}\left(\boldsymbol{F}^{\mathrm{T}}\boldsymbol{F}\right)^{-1}\boldsymbol{F}^{\mathrm{T}}\bar{\boldsymbol{y}} = \frac{1}{\Delta t}\boldsymbol{F}^{+}\bar{\boldsymbol{y}}. \tag{3}$$

For measurements, multiple impacts can be averaged using the time domain equivalent of the $H_1$-FRF estimator. Vertically stacking the force convolution matrices $\boldsymbol{F}$ of each impact in $\boldsymbol{F}_{\text{stacked}}$ allows to calculate an averaged auto-correlation matrix $\boldsymbol{F}_{\text{stacked}}^{\mathrm{T}}\boldsymbol{F}_{\text{stacked}}$. Analogously, the responses can be vertically stacked in an array $\bar{\boldsymbol{y}}_{\text{stacked}}$, and the averaged cross-correlation then follows as $\boldsymbol{F}_{\text{stacked}}^{\mathrm{T}}\bar{\boldsymbol{y}}_{\text{stacked}}$. Building the ratio between the cross-correlation and the auto-correlation yields the averaged IRF $\bar{\boldsymbol{h}}^{\text{avg}}$:

$$\bar{\boldsymbol{h}}^{\text{avg}} = \frac{1}{\Delta t}\left(\boldsymbol{F}_{\text{stacked}}^{\mathrm{T}}\boldsymbol{F}_{\text{stacked}}\right)^{-1}\boldsymbol{F}_{\text{stacked}}^{\mathrm{T}}\bar{\boldsymbol{y}}_{\text{stacked}} = \frac{1}{\Delta t}\boldsymbol{F}_{\text{stacked}}^{+}\bar{\boldsymbol{y}}_{\text{stacked}}. \tag{4}$$

### *Comparison of Signals in the Time Domain*

To evaluate the differences between two discrete time domain signals, the metrics defined in the following are used. A reference IRF for this can be, for instance, a numerically generated IRF $h_{\text{num}}[k]$ from a Newmark time integration with proper initial conditions, see [5] for details. However, this then mainly evaluates the difference to the numerical model, rather than if the selected IRF length $t_{\text{IRF}}$ is already sufficient to extract all relevant information from the given measurements. Therefore, in the following, the IRFs are compared to an IRF $h_{\text{exp}}[k]$ calculated with a longer estimation length $t_{\text{IRF}}$. When the length $t_{\text{IRF}}$ is chosen sufficiently large, i.e., more data will not improve the IRF anymore, the comparison directly shows whether the IRF being compared is already converged to the best possible one. Compared are only the values of the IRFs until $t_{\text{IBS}}$, the amount that will later be used in the IBS scheme, since only this part of the IRFs is of interest.

**Comparison of Signal Energy** The first method considered is to compare the signal energy (not a mechanical energy) of the IRFs $h[k]$. In general, the energy of a discrete signal is given by [7]:

$$E(h) = \sum_{k=0}^{n} |h[k]|^2 . \tag{5}$$

Then, the energy difference $\Xi_{\mathrm{E}}$ between the reference IRF $h_{\mathrm{ref}}[k]$ and the IRF being compared $h_{\mathrm{comp}}[k]$ can be found with:

$$\Xi_{\mathrm{E}}(h_{\mathrm{comp}}, h_{\mathrm{ref}}) = \frac{E(h_{\mathrm{comp}})}{E(h_{\mathrm{ref}})} - 1. \tag{6}$$

**Comparison of Time Domain Shape** Another option is to compare the shape of both signals by looking at the absolute difference at each discrete time step $k$. The time domain difference metric $\Xi_{\mathrm{TDS}}$ is additionally normalized with the maximum value of the reference signal $h_{\mathrm{ref}}[k]$:

$$\Xi_{\mathrm{TDS}}(h_{\mathrm{comp}}, h_{\mathrm{ref}}) = \frac{\sum_{k=0}^{n} |h_{\mathrm{comp}}[k] - h_{\mathrm{ref}}[k]|}{\max(h_{\mathrm{ref}})}. \tag{7}$$

One drawback of this method is that time shifts or slightly wrong frequencies can lead to huge differences since this comparison does not take into account the frequency content. This is problematic when numerical IRFs $h_{\mathrm{num}}[k]$ are used because the eigenfrequencies will most likely not match the experimental ones. However, this is not a problem when IRFs with different estimation lengths $t_{\mathrm{IRF}}$ are compared since the frequency content should be identical.

## *Criteria for Optimal IRF Length*

The metrics introduced in the previous subsection can be used to assess the quality of the estimated IRFs. However, the IRF $h_{\mathrm{comp}}[k]$ best matching the reference IRF $h_{\mathrm{ref}}[k]$ will most likely be the one with the highest estimation length $t_{\mathrm{IRF}}$ that uses the most input data, which ignores the greatly increasing computations costs. Therefore, the computational costs, which are proportional to the length $t_{\mathrm{IRF}}$, should also be considered. Especially since only the initial part of the IRF, for instance, around $2\,\mathrm{ms}$, will be used for IBS, the usefulness of additional data will diminish at some point. To assess this problem, the Akaike Information Criterion and the Bayesian Information Criterion are introduced in the following.

**Akaike Information Criterion (AIC)** The AIC provides a measure of goodness for statistical models and has the lowest values for the best fitting model. It is, in general, defined as [8–11]:

$$\mathrm{AIC} = 2k - 2\log(\hat{L}), \tag{8}$$

where $k$ is the model size and $\hat{L}$ is the maximum likelihood function (MLF). The MLF $\hat{L}$ indicates how well the determined model fits the available data and is maximal for the best one [10]. For IRFs, the quality of the identified IRF $\bar{\boldsymbol{h}}$ can, for example, be judged by calculating the residual of the Time Domain Deconvolution definition in eq. (2):

$$\mathrm{res}_h = \left\| \boldsymbol{F}\bar{\boldsymbol{h}}\Delta t - \bar{\boldsymbol{y}} \right\| . \tag{9}$$

This residual is further normalized by dividing by $\bar{\boldsymbol{y}}[k]$ at each discrete time $k$ and the number of data points $n_{\mathrm{res}}$ in the residual. Since the quality of the IRF is only important for the amount of time used in the IBS scheme $t_{\mathrm{IBS}}$, the residual is only evaluated for this interval, i.e., $n_{\mathrm{res}}$ is the number of points in the residual evaluated until $t_{\mathrm{IBS}}$:

$$\mathrm{res}_h\Big|_{t=0}^{t_{\mathrm{IBS}}} = \left\| \sum_{k=0}^{n_{\mathrm{res}}} \left( \frac{\bar{\boldsymbol{y}}_h[k] - \bar{\boldsymbol{y}}[k]}{\bar{\boldsymbol{y}}[k]} \right) \right\| \cdot \frac{1}{n_{\mathrm{res}}} \qquad \text{with} \qquad \bar{\boldsymbol{y}}_h = \boldsymbol{F}\bar{\boldsymbol{h}}\Delta t. \tag{10}$$

However, the residual $\mathrm{res}_h\big|_{t=0}^{t_{\mathrm{IBS}}}$ becomes minimal for the best model, not maximal as required for $\hat{L}$. To accommodate this, the sign of the $2\log(\hat{L})$ term in eq. (8) is changed. Further substituting in eq. (8) the model size $k$ with the IRF size $n_{\mathrm{IRF}}$, yields the $\mathrm{AIC}_{\mathrm{res}_h}$ adjusted for IRFs calculated through Time Domain Deconvolution:

$$\mathrm{AIC}_{\mathrm{res}_h} = 2n_{\mathrm{IRF}} + 2\log\left(\mathrm{res}_h\Big|_{t=0}^{t_{\mathrm{IBS}}}\right). \tag{11}$$

To summarize, the adjusted $\mathrm{AIC}_{\mathrm{res}_h}$ is minimal for the best combination of small residual $\mathrm{res}_h\big|_{t=0}^{t_{\mathrm{IBS}}}$ within the IBS evaluation time $t_{\mathrm{IBS}}$ and low model size $n_{\mathrm{IRF}}$.

**Bayesian Information Criterion (BIC)** The BIC is given in general as [10–12]:

$$\mathrm{BIC} = k\log(n) - 2\log(\hat{L}), \tag{12}$$

which has an additional parameter $n$, the number of independent observations [11]. For IRFs, this parameter corresponds to the number of averages $n_{\mathrm{avg}}$. With the same adjustments as for the AIC, the adjusted $\mathrm{BIC}_{\mathrm{res}_h}$ using the residual $\mathrm{res}_h\big|_{t=0}^{t_{\mathrm{IBS}}}$ becomes:

$$\mathrm{BIC}_{\mathrm{res}_h} = n_{\mathrm{IRF}}\log(n_{\mathrm{avg}}) + 2\log\left(\mathrm{res}_h\big|_{t=0}^{t_{\mathrm{IBS}}}\right). \tag{13}$$

The $\mathrm{BIC}_{\mathrm{res}_h}$ also is minimal for the best IRF estimate, but also considers the number of averages $n_{\mathrm{avg}}$. This difference is generally important because $n_{\mathrm{avg}}$ is not present in the $\mathrm{AIC}_{\mathrm{res}_h}$, however, having different observations or impacts rather than just more time data from the same observation will be beneficial.

## Experimental Study

In the following, the criteria for optimal IRF length $t_{\mathrm{IRF}}$ described in the theory section are applied to an experimental test case. First, the experimental setup is described, then the IRF criteria are applied to the experimental data using different amount of averages. The optimal IRF lengths are then used for IBS, and the difference compared to IRFs with significantly longer estimation length is evaluated.

### *Experimental Setup*

The test case consists of three aluminum rods, see also fig. 3: Substructure $S_1$ with length $\ell_1 = 300\,\mathrm{mm}$ and Substructure $S_2$ with length $\ell_2 = 600\,\mathrm{mm}$, that shall be assembled to the reference system $S_0$ using IBS. To provide a reference, $S_0$ is also measured. The rods are suspended in an approximate 'free-free' boundary condition using foam blocks. Using an impact hammer with a steel tip, imperfect impulses are applied to the system, and the acceleration responses $a_i^{(s)}$ are measured. Using a sample rate of $102.4\,\mathrm{kHz}$, 20 impacts are measured. More details about the measurement setup can be found in [1]; the raw measurement data is available for download in [13].

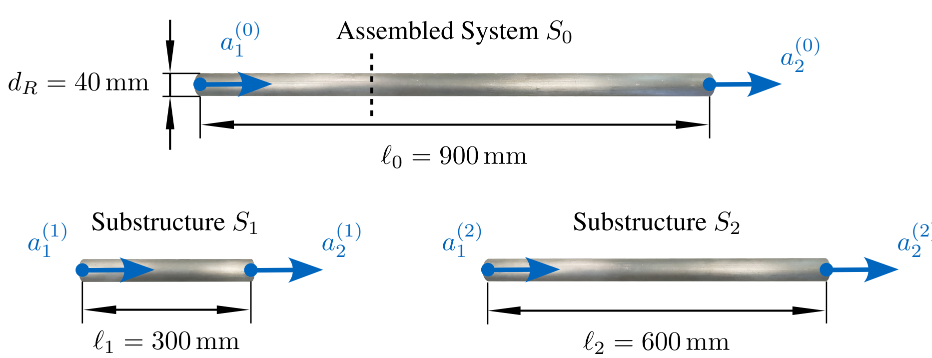


**Fig. 3** Overview of one-dimensional rods used as test pieces in experimental study

### *Evaluation of IRF Length Criteria*

In total, four IRFs are calculated: Two driving point IRFs $h_{11}^{(s)}$ and two cross point IRFs $h_{21}^{(s)}$, one for each substructure $s$. The other IRFs required for IBS are found using the symmetry of the system. First, the difference of signal energy $\Xi_{\mathrm{E}}$, the difference of time domain shape $\Xi_{\mathrm{TDS}}$, the adjusted Akaike Information Criterion $\mathrm{AIC}_{\mathrm{res}_h}$, and the adjusted Bayesian Information Criterion $\mathrm{BIC}_{\mathrm{res}_h}$ are evaluated using three impacts of the available measurements. For this, reference IRFs are calculated with length $t_{\mathrm{IRF}} = 200\,\mathrm{ms}$, against which IRFs using the same data but with varying length, from 2.1 to $200\,\mathrm{ms}$, are compared. Figure 4 shows the respective results for the individual IRFs as well as the sum over all IRFs, to give a global indicator.

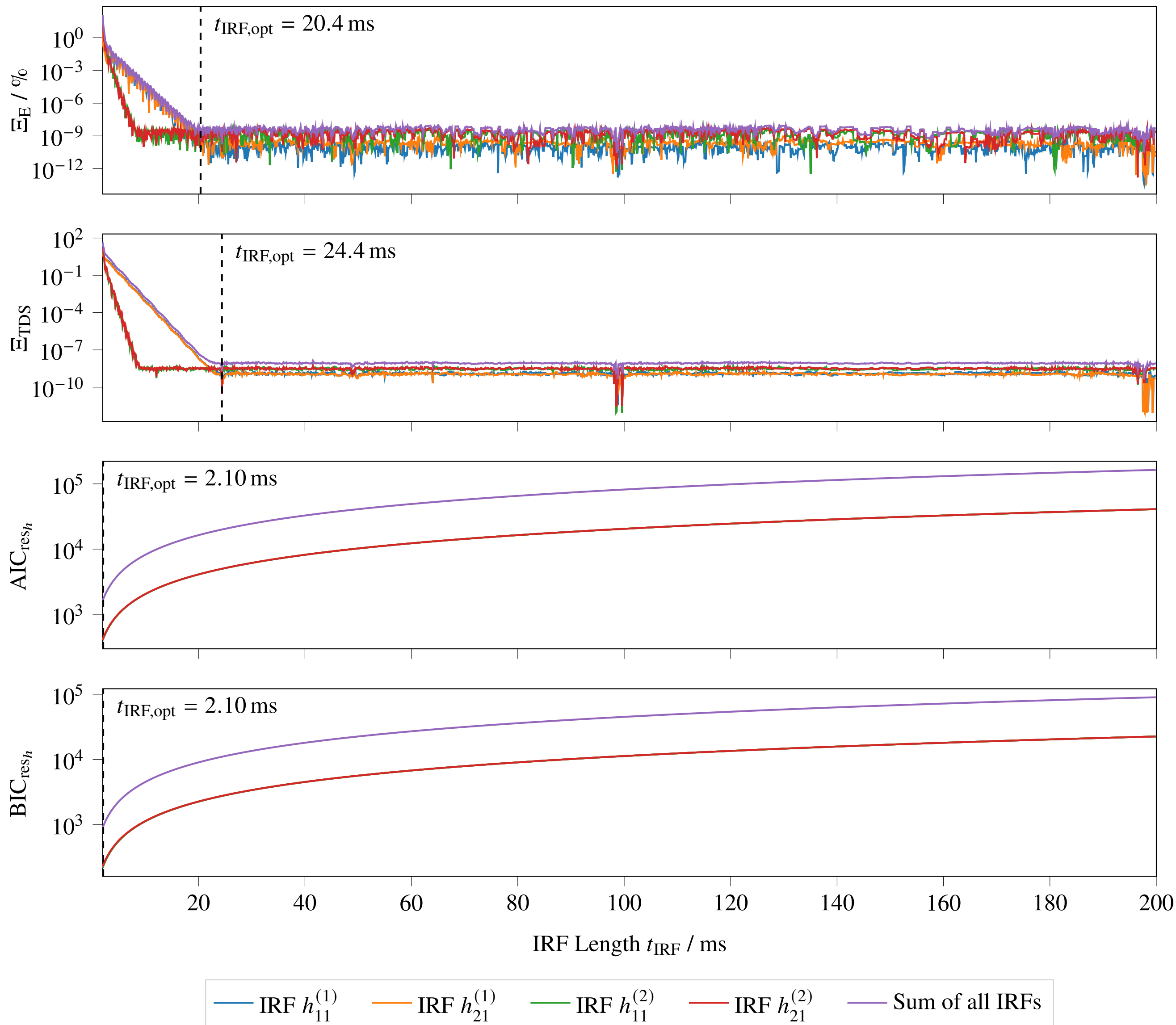


**Fig. 4** IRF criteria results from **experimental** study using **3 averages**, compared to experimental IRF from the same data with $t_{\mathrm{IRF}} = 200\,\mathrm{ms}$. Used criteria: $\Xi_{\mathrm{E}}$ – difference of signal energy, $\Xi_{\mathrm{TDS}}$ – difference of time domain shape, $\mathrm{AIC}_{\mathrm{res}_h}$ – Akaike Information Criterion for IRFs, $\mathrm{BIC}_{\mathrm{res}_h}$ – Bayesian Information Criterion for IRFs

As can be seen, the error indicated by both the comparison of signal energy $\Xi_{\mathrm{E}}$ and of the time domain shape $\Xi_{\mathrm{TDS}}$ is steadily decreasing for longer IRF estimation lengths $t_{\mathrm{IRF}}$. The IRFs of substructure 2 are thereby converging faster than the ones of substructure 1. For both criteria, a point can be determined from the plot after which no significant improvement is achieved by adding more data. This point could now be determined for each IRF individually or, for convenience, using the sum to get one optimal IRF length $t_{\mathrm{IRF,opt}}$ for all IRFs. Looking at the sum over all IRFs for $\Xi_{\mathrm{E}}$, this length is determined as $t_{\mathrm{IRF,opt}} = 20.4\,\mathrm{ms}$. Using $\Xi_{\mathrm{TDS}}$, the optimal length is $t_{\mathrm{IRF,opt}} = 24.4\,\mathrm{ms}$.

Both the $\mathrm{AIC}_{\mathrm{res}_h}$ and $\mathrm{BIC}_{\mathrm{res}_h}$ suggest that the optimal IRF length is $t_{\mathrm{IRF,opt}} = 2.1\,\mathrm{ms}$, i.e., the smallest possible length for an IBS evaluation of $t_{\mathrm{IBS}} = 2.0\,\mathrm{ms}$. Whether the estimated IRFs are then suitable for IBS will be evaluated in the next subsection. The reason for this is that both criteria are dominated by the term containing $n_{\mathrm{IRF}}$. In contrast, the residual term is relatively small in magnitude, e.g., for a residual of $10^{-8}$, $\log 10^{-8} \approx -18.42$. Further, the residual (eq. (10)) of the Time Domain Deconvolution (eq. (2)) will most likely not converge to zero when noise and other experimental errors are present. For instance, the residual norms (eq. (10)) for $t_{\mathrm{IRF}} = 200\,\mathrm{ms}$ are: $\mathrm{res}_{h_{11}^{(1)}} \approx 0.2052$, $\mathrm{res}_{h_{21}^{(1)}} \approx 2.173$, $\mathrm{res}_{h_{11}^{(2)}} \approx 0.0172$, and $\mathrm{res}_{h_{21}^{(2)}} \approx 3.395$. This means that changes in the residual will barely have any effect on the $\mathrm{AIC}_{\mathrm{res}_h}$ or $\mathrm{BIC}_{\mathrm{res}_h}$.

One option to make the $\mathrm{AIC}_{\mathrm{res}_h}$ and $\mathrm{BIC}_{\mathrm{res}_h}$ more meaningful would be to tune the ratio between the computational cost and the residual term. For example, the logarithm could be replaced by $1/x$ to increase weight when the residual is close to zero. Since the residual might also not get close to zero, the residual could be replaced by $\Xi_{\mathrm{TDS}}$, which converges to the reference IRF. However, the question arises whether the original intention of the criteria is preserved and how the new parameters and/or functions should be chosen. For those reasons, the criteria are not further adapted here.

The results using five impacts are shown in fig. 5. While $\mathrm{AIC}_{\mathrm{res}_h}$ and $\mathrm{BIC}_{\mathrm{res}_h}$ show the same qualitative results, i.e., $t_{\mathrm{IRF,opt}} = 2.1\,\mathrm{ms}$, the time domain metrics show a difference: For both criteria, it can be seen that the calculated IRFs converge faster to the reference IRF. Looking at the sum of each criteria, both identify the optimal length as $t_{\mathrm{IRF,opt}} = 15.3\,\mathrm{ms}$. Using 10 or 20 averages lets the IRFs converge even faster, see fig. 8 and fig. 9 in the appendix. For 10 averages $\Xi_{\mathrm{E}}$ and $\Xi_{\mathrm{TDS}}$ yield $t_{\mathrm{IRF,opt}} = 3.4\,\mathrm{ms}$, for 20 averages $t_{\mathrm{IRF,opt}} = 3.3\,\mathrm{ms}$. In the next section, the suitability of the identified IRF lengths for different amounts of averages is tested within the IBS scheme.

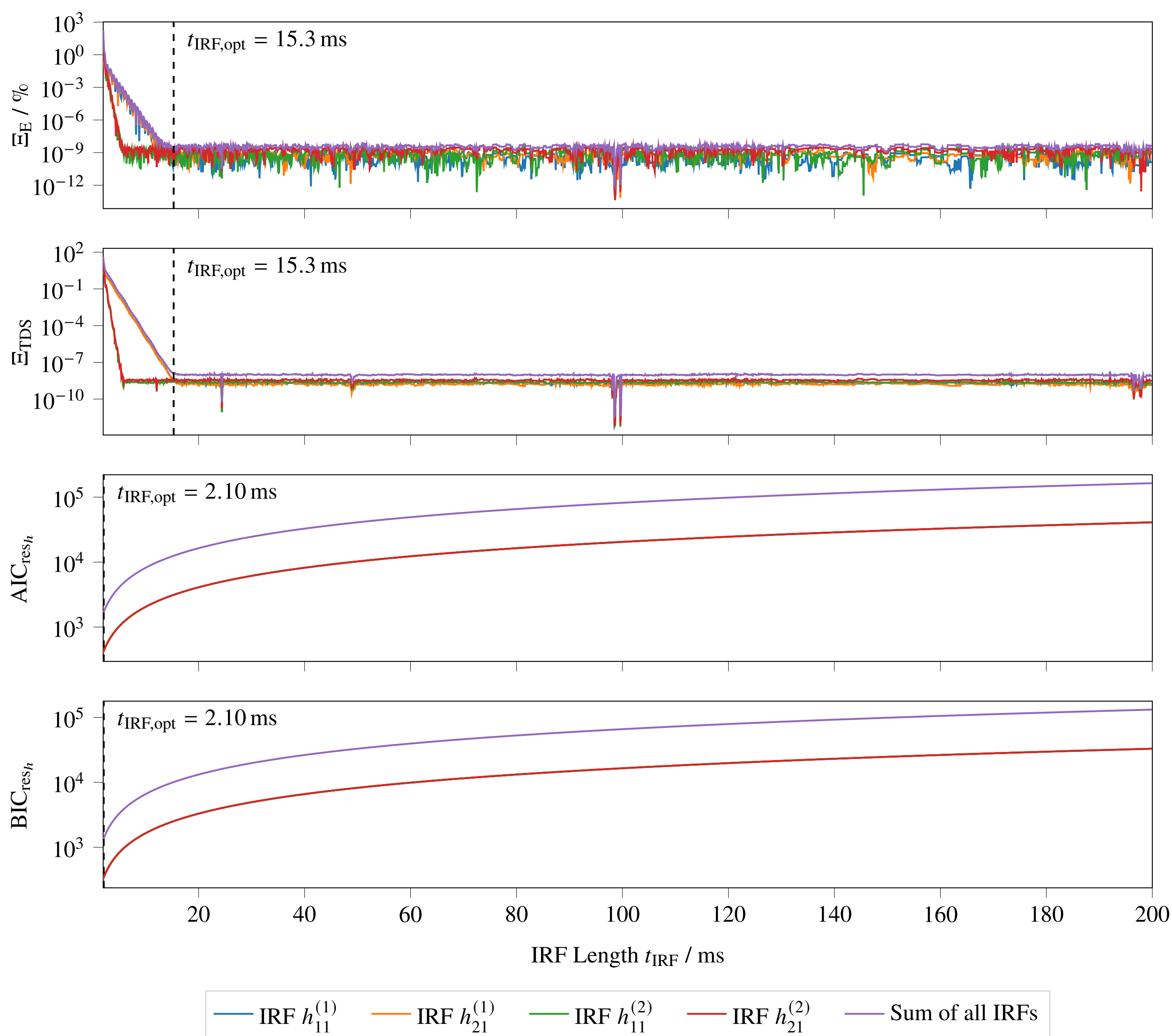


**Fig. 5** IRF criteria results from **experimental** study using **5 averages**, compared to experimental IRF from the same data with $t_{\mathrm{IRF}} = 200\,\mathrm{ms}$. Used criteria: $\Xi_{\mathrm{E}}$ – difference of signal energy, $\Xi_{\mathrm{TDS}}$ – difference of time domain shape, $\mathrm{AIC}_{\mathrm{res}_h}$ – Akaike Information Criterion for IRFs, $\mathrm{BIC}_{\mathrm{res}_h}$ – Bayesian Information Criterion for IRFs

### *IBS Results using Determined Optimal IRF Lengths*

The determined optimal IRF lengths $t_{\text{IRF}}$ are summarized in table 1. To evaluate the quality of the IRFs, the IBS scheme is evaluated with $t_{\text{IRF}} = 50\,\text{ms}$ and with the determined optimal length $t_{\text{IRF,opt}}$. The generated responses of the assembled system $S_0$ are then compared using the same metrics as introduced for the IRFs, the difference in signal energy $\Xi_{\text{E}}$ and the difference in time domain shape $\Xi_{\text{TDS}}$. As can be seen, when the IRF lengths determined through $\Xi_{\text{E}}$ and $\Xi_{\text{TDS}}$ are used, there is no detectable difference compared to $t_{\text{IRF}} = 50\,\text{ms}$. This means the IRF length can be significantly reduced compared to previous works [1, 2, 14] (always using $t_{\text{IRF}} = 50\,\text{ms}$), for instance, by one order of magnitude for $n_{\text{avg}} = 20$. The computational costs can be decreased by a factor of $\approx 15.7$ when an energy error of $0.0005\,\%$ is accepted.

**Table 1** Summary of optimal IRF length $t_{\text{IRF,opt}}$ determined through difference of signal energy $\Xi_{\text{E}}$, difference of time domain shape $\Xi_{\text{TDS}}$, Akaike Information Criterion for IRFs $\text{AIC}_{\text{res}_h}$, Bayesian Information Criterion for IRFs $\text{BIC}_{\text{res}_h}$, and manual tuning. The IRF Quality is assessed by evaluating IBS with $t_{\text{IRF}} = 50\,\text{ms}$ and the respective $t_{\text{IRF,opt}}$, and comparing the signal energy ($\Xi_{\text{E}}$) and time domain shape ($\Xi_{\text{TDS}}$) of the calculated response

| **Averages** | **Determined IRF Length** | | $\Xi_{\text{E,IBS}}$ ($t_{\text{IRF}} = 50\,\text{ms}, t_{\text{IRF,opt}}$) | | $\Xi_{\text{TDS,IBS}}$ ($t_{\text{IRF}} = 50\,\text{ms}, t_{\text{IRF,opt}}$) | | **Computation Time** |
|---|---|---|---|---|---|---|---|
| | Method | $t_{\text{IRF,opt}}$ | $a_1^{(0)}$ | $a_2^{(0)}$ | $a_1^{(0)}$ | $a_2^{(0)}$ | (Averaged, per IRF) |
| 3 averages | Reference IRF | 50.0 ms | – | – | – | – | 451.6 ms (=100 %) |
| | $\Xi_{\text{E}}$ | 20.4 ms | 0.0000 % | 0.0000 % | 0.0000 | 0.0000 | 184.1 ms (≈ 40.8 %) |
| | $\Xi_{\text{TDS}}$ | 24.4 ms | 0.0000 % | 0.0000 % | 0.0000 | 0.0000 | 216.5 ms (≈ 48.0 %) |
| | Manual Tune | 5.00 ms | 0.4759 % | 0.1160 % | 0.1555 | 0.1352 | 42.65 ms (≈ 9.45 %) |
| | $\text{AIC}_{\text{res}_h}$ & $\text{BIC}_{\text{res}_h}$ | 2.10 ms | −0.5756 % | 3.450 % | 2.259 | 3.981 | 20.43 ms (≈ 4.52 %) |
| 5 averages | Reference IRF | 50.0 ms | – | – | – | – | 827.0 ms (=100 %) |
| | $\Xi_{\text{E}}$ & $\Xi_{\text{TDS}}$ | 15.3 ms | 0.0000 % | 0.0000 % | 0.0000 | 0.0000 | 222.0 ms (≈ 26.8 %) |
| | Manual Tune | 3.00 ms | 0.5640 % | −0.0332 % | 0.3492 | 0.5402 | 43.25 ms (≈ 5.23 %) |
| | $\text{AIC}_{\text{res}_h}$ & $\text{BIC}_{\text{res}_h}$ | 2.10 ms | 2.215 % | 0.8929 % | 1.495 | 2.777 | 32.63 ms (≈ 3.95 %) |
| 10 averages | Reference IRF | 50.0 ms | – | – | – | – | 1424 ms (=100 %) |
| | $\Xi_{\text{E}}$ & $\Xi_{\text{TDS}}$ | 5.90 ms | 0.0000 % | 0.0000 % | 0.0000 | 0.0000 | 177.6 ms (≈ 12.5 %) |
| | Manual Tune | 3.00 ms | 0.0039 % | −0.0020 % | 0.0076 | 0.0053 | 87.73 ms (≈ 6.16 %) |
| | $\text{AIC}_{\text{res}_h}$ & $\text{BIC}_{\text{res}_h}$ | 2.10 ms | 3.237 % | −0.2112 % | 1.537 | 1.636 | 63.35 ms (≈ 4.45 %) |
| 20 averages | Reference IRF | 50.0 ms | – | – | – | – | 2936 ms (=100 %) |
| | $\Xi_{\text{E}}$ & $\Xi_{\text{TDS}}$ | 5.00 ms | 0.0000 % | 0.0000 % | 0.0000 | 0.0000 | 312.6 ms (≈ 10.6 %) |
| | Manual Tune | 3.00 ms | 0.0005 % | 0.0001 % | 0.0013 | 0.0010 | 187.3 ms (≈ 6.38 %) |
| | $\text{AIC}_{\text{res}_h}$ & $\text{BIC}_{\text{res}_h}$ | 2.10 ms | 2.507 % | −0.0639 % | 1.554 | 1.195 | 154.0 ms (≈ 5.25 %) |

Using an IRF length of $t_{\text{IRF}} = 2.1\,\text{ms}$, as identified by $\text{AIC}_{\text{res}_h}$ and $\text{BIC}_{\text{res}_h}$ for all $n_{\text{avg}}$, gives errors in the signal energy of up to $3.450\,\%$. The worst results are achieved for $n_{\text{avg}} = 3$, for which the IRF results are shown in fig. 6.

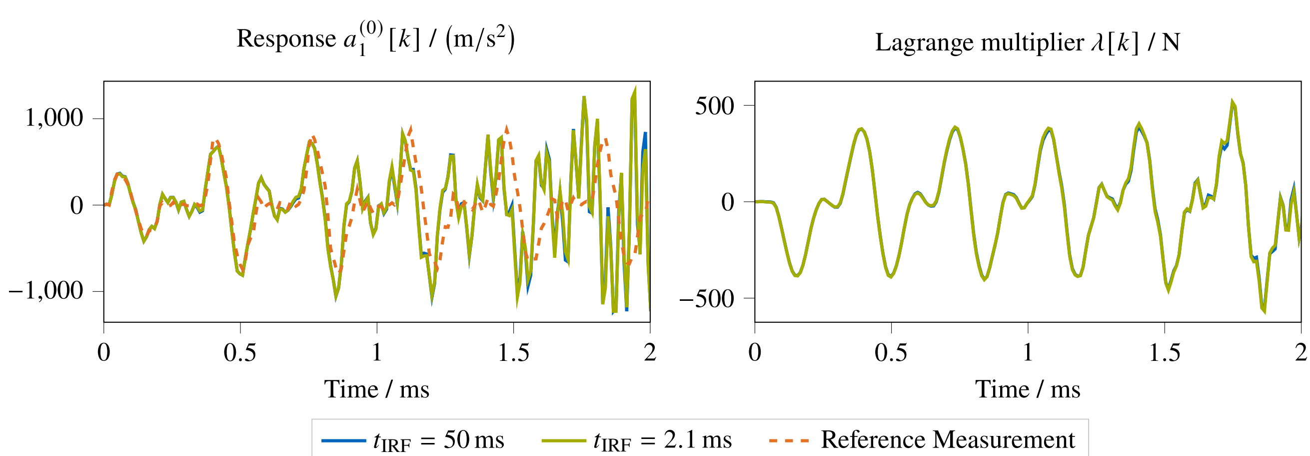


**Fig. 6** Comparison of experimental IBS results with three averages using $t_{\text{IRF}} = 50\,\text{ms}$ and $t_{\text{IRF}} = 2.1\,\text{ms}$. As a reference, a measurement of the assembled system's response to the force used in the IBS scheme is given

As can be seen, the IRF results do not differ significantly; however, there are some bigger errors in the amplitudes, especially toward the end of the response. Whether those kinds of errors are tolerable would depend on the use case; the stability of the IBS scheme is not affected here. When the IRF length is adjusted manually, a better compromise between calculation time and accuracy can be achieved, e.g., using $t_{\mathrm{IRF}} = 5\,\mathrm{ms}$ for $n_{\mathrm{avg}} = 3$ yields a maximum signal energy error of $0.4759\,\%$. Especially for $n_{\mathrm{avg}} = 10$ and $n_{\mathrm{avg}} = 20$, using an adjusted value of $t_{\mathrm{IRF}} = 3\,\mathrm{ms}$ yields negligible errors, while further decreasing the calculation time by approximately $40\,\%$.

It is worth noting that even when the optimal IRF length is used, the IBS results may still be suboptimal. Looking at the IBS results with $t_{\mathrm{IRF}} = 50\,\mathrm{ms}$ and a varying number of averages in fig. 7, better results, i.e., with reduced instability toward the end, are achieved with a higher number of averages, e.g., with $n_{\mathrm{avg}} = 10$ or $n_{\mathrm{avg}} = 20$.

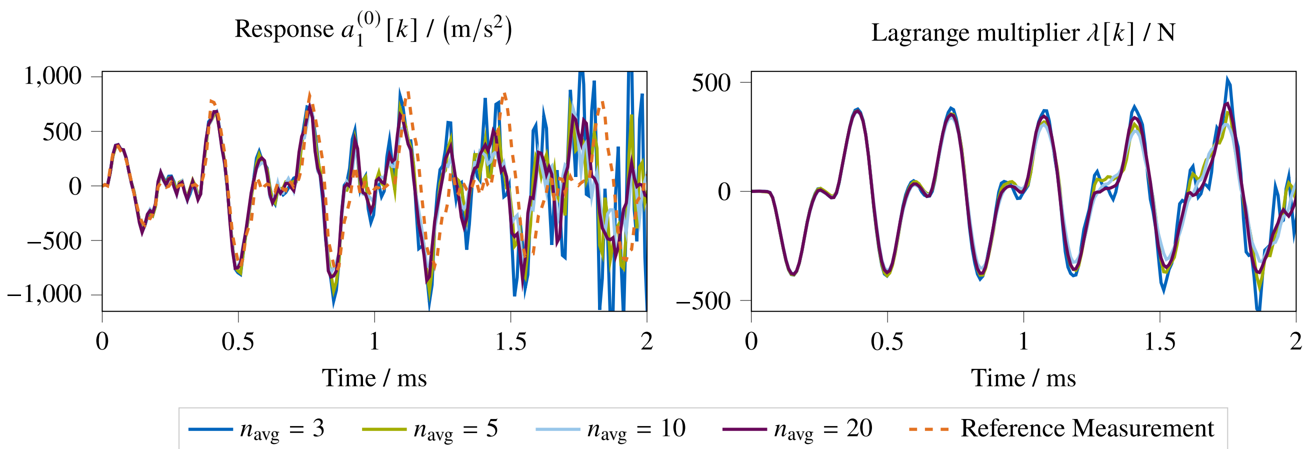


**Fig. 7** Comparison of IBS results with different number of averages as labeled. Results using optimal IRF length from $\Xi_{\mathrm{E}}$ or $\Xi_{\mathrm{TDS}}$ and $t_{\mathrm{IRF}} = 50\,\mathrm{ms}$ are identical; only one result is shown for each number of averages

## Summary and Conclusion

In this paper, criteria for determining the optimal IRF length $t_{\mathrm{IRF}}$ were evaluated. Two criteria, the difference of the signal energy $\Xi_{\mathrm{E}}$ and the difference of the time domain shape $\Xi_{\mathrm{TDS}}$, were used to compare IRFs with varying length $t_{\mathrm{IRF}}$ against a reference IRF with $t_{\mathrm{IRF}} = 200\,\mathrm{ms}$, 100 times the length of the desired IBS evaluation time of $t_{\mathrm{IBS}} = 2\,\mathrm{ms}$. From the plots, the point at which the IRF converged to the reference IRF could be determined. Using this length yielded IBS results with no detectable difference compared to $t_{\mathrm{IRF}} = 200\,\mathrm{ms}$, or the length of $t_{\mathrm{IRF}} = 50\,\mathrm{ms}$ used in previous works [1, 2, 14]. This reduction in length is motivated by a reduction in computational costs. For $n_{\mathrm{avg}} = 20$, the computation costs could be reduced by a factor of ten. This is especially important for 3D cases, like the one described in [14], where a total of 1254 IRFs must be calculated. Nevertheless, an open question is how this check of IRF convergence would be implemented efficiently in an automatic IRF calculation algorithm.

Using the Akaike Information Criterion for IRFs $\mathrm{AIC}_{\mathrm{res}_h}$ and the Bayesian Information Criterion for IRFs $\mathrm{BIC}_{\mathrm{res}_h}$, a signal energy error of up to $3.450\,\%$ was found. Both criteria always suggested the lowest possible IRF length $t_{\mathrm{IRF}}$. It is questionable whether the criteria as used provide valuable information. Both criteria's balance between model size and model fit could be adjusted to favor better fitting models and place less weight on the model size.

On the other hand, those results might also indicate that directly fitting the time series values of the IRFs to the measured data is a bad idea, and, instead, it might be better to fit a suitable mechanical model with 10-20 degrees of freedom to the data. Calculating the pseudo-inverse $\boldsymbol{F}^{+}$ becomes problematic when the condition number is large, or, in other words, when columns of the matrix are nearly linearly dependent. This can be an indicator of multicollinearity, meaning that some of the variables in the regression model, here the values of the IRF's time series, are highly correlated [15, 16]. One symptom of multicollinearity is a residual that is not steadily decreasing when more data is added. Another option for detecting multicollinearity is to use the variance inflation factor (VIF) [15], which should be explored in the future.

With the presented methods, it is possible to judge how much measurement data is required to get the best IRF estimate, without needing a reference model. Using the difference of the signal energy $\Xi_{\mathrm{E}}$ or the difference of the time domain shape $\Xi_{\mathrm{TDS}}$, it can be checked if the IRF estimation length was chosen long enough. However, this does not guarantee that the IRF estimate is a good estimate of the true IRF nor that an evaluation of the IBS scheme will perform well.

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## Appendix: Additional IRF Criteria Evaluation Results

In the following, the IRF criteria results for 10 (fig. 8) and 20 averages (fig. 9) are shown. Note that, compared to the results for 3 averages (fig. 4) and 5 averages (fig. 5), the IRF length $t_{\mathrm{IRF}}$ is only displayed until $100\,\mathrm{ms}$.

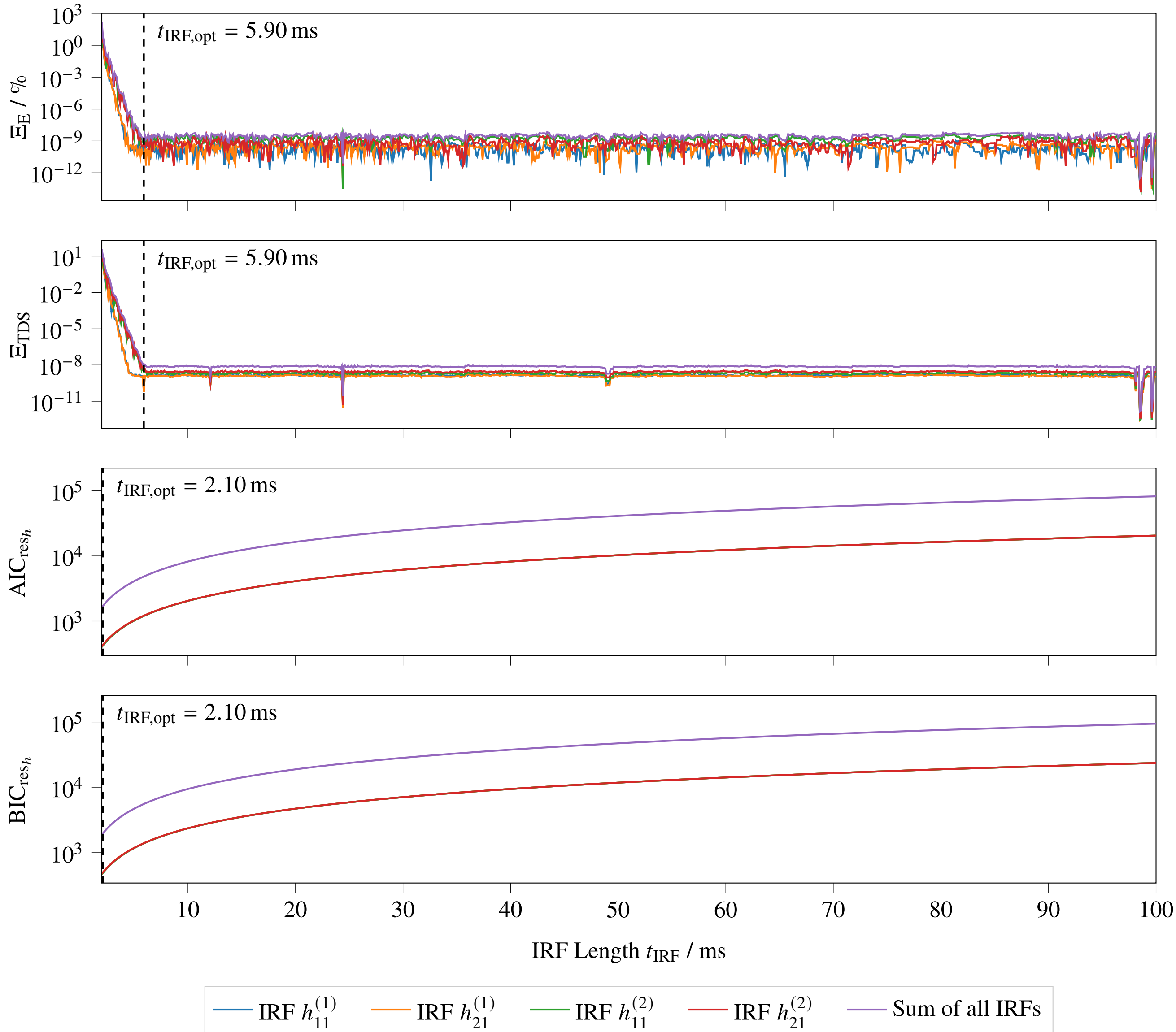


**Fig. 8** IRF criteria results from **experimental** study using **10 averages**, compared to experimental IRF from the same data with $t_{\mathrm{IRF}} = 200\,\mathrm{ms}$. Used criteria: $\Xi_{\mathrm{E}}$ – difference of signal energy, $\Xi_{\mathrm{TDS}}$ – difference of time domain shape, $\mathrm{AIC}_{\mathrm{res}_h}$ – Akaike Information Criterion for IRFs, $\mathrm{BIC}_{\mathrm{res}_h}$ – Bayesian Information Criterion for IRFs

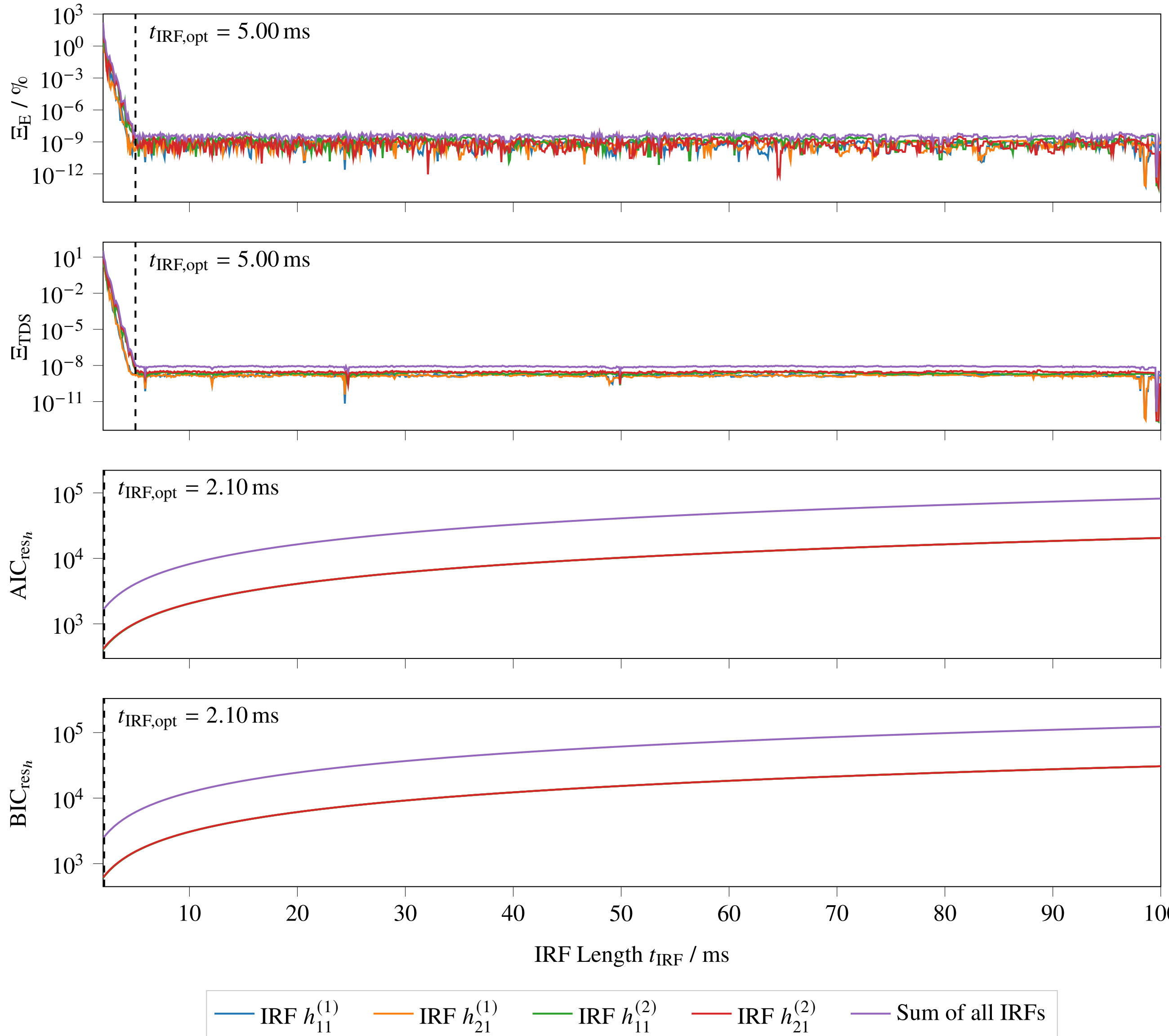


**Fig. 9** IRF criteria results from **experimental** study using **20 averages**, compared to experimental IRF from the same data with $t_{\mathrm{IRF}} = 200\,\mathrm{ms}$. Used criteria: $\Xi_{\mathrm{E}}$ – difference of signal energy, $\Xi_{\mathrm{TDS}}$ – difference of time domain shape, $\mathrm{AIC}_{\mathrm{res}_h}$ – Akaike Information Criterion for IRFs, $\mathrm{BIC}_{\mathrm{res}_h}$ – Bayesian Information Criterion for IRFs