Chapter 2
A Spectral Matrix Estimation Scheme for Pseudorandom Excitation with Concurrent Phase Combinations

Kevin Napolitano and William Fladung

Abstract Pseudorandom is an excitation technique that allows the user to precisely define source signal auto-spectral density values by assigning a sinusoidal magnitude and phase at the center of each frequency bin. It is used in single-degree-of-freedom environmental testing and modal testing. In order to extend the method to multiple sources for modal testing, sequential sets of varying phase combinations must be applied. This paper shows how combining Daniell spectral estimation with optimal phasing assignment at each specified frequency can result in a precisely defined source cross-spectral density matrix that can help extend pseudorandom to multiple-degree-of-freedom environmental testing and also has the potential to be used for operational modal analysis that uses cross-spectrum directly to estimate modes.

Keywords Spectral estimation · Modal testing · Environmental testing · Daniell · Pseudorandom

Introduction

Pseudorandom is an excitation technique that allows the user to precisely define source signal auto-spectral density values by assigning the magnitude and phase of a sinusoidal input at the center of each frequency bin [1]. Typically, a pseudorandom source signal is generated from a spectrum with the associated frequency spacing and is repeated for several frames. After transients decay, the steady-state response satisfies the assumption of the discrete Fourier transform (DFT) that the signals are periodic. In order to extend pseudorandom excitation to multiple-reference modal tests, sequential tests must be performed with each test having a different phase combination between the sources at every frequency.

The main advantages of pseudorandom excitation are that it is a leakage-free technique that requires no windows and allows the user to specify a precise level at each frequency. Single-axis shaker controllers can exploit this feature to precisely define input auto-spectrum at all required frequencies. The main disadvantage of pseudorandom excitation for modal testing is that, once data is collected, there is no way to increase the frequency resolution by altering signal processing parameters. Also, time is taken between each source phase change as the structure settles into a steady-state response.

While this method can result in very clean frequency response functions (FRFs), it is not useful for multiple-reference random shaker control because the control signal would need to change in the middle of a test with each phasing combination.

This paper presents a method to generalize the pseudorandom excitation method for multiple sources by exploiting the Daniell spectral estimation method [2] to apply the different source phasing combinations concurrently. Doing so can help provide a uniform excitation over the course of a test and can minimize data collection times. Combined with optimal phasing assignment [3], off-diagonals of a source signal cross-spectrum matrix can be precisely defined, and therefore the response auto-spectrum matrix can also be precisely defined, which allows for the precision of pseudorandom excitation to be extended to environmental testing as well as allows for modal analysis of tests with very large numbers of references, such as those associated with acoustic excitation [4], [5], where the calculation of FRFs can be burdensome.

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Concurrent Phase Combinations Using Daniell Spectral Estimation

Bartlett spectral estimation [6] involves performing a DFT on a several frames of time-history data and then taking the average of their respective cross-spectral matrices, such that

$$[S_{WZ}(\omega_i)] = (1/N)\sum_{k=1}^{N}\{w_k(\omega_i)\}\{z_k(\omega_i)\}^H \tag{1}$$

where $\{w_k(\omega_i)\}$ and $\{z_k(\omega_i)\}$ are the DFT of the "kth" frame of data for two sets of channels at a given frequency (ω_i), and N is the total number of frames of data. At its fundamental level, Daniell spectral estimation involves calculating a DFT of the entire time history and summing the spectral lines near the frequency of interest, such that

$$[S_{WZ}(\omega_i)] = \left(\frac{1}{2m+1}\right)\sum_{k=i-m}^{i+m}\{w(\omega_k)\}\{z(\omega_k)\}^H \tag{2}$$

where $\{w(\omega_k)\}$ and $\{z(\omega_k)\}$ are the DFT of the kth frequency bin for two sets of channels, and m is the number of frequency bins to the left and right of the frequency of interest.

As with traditional pseudorandom excitation, a sine wave is generated at the center of each frequency bin, and therefore a window is not needed to reduce leakage since the assumption of periodicity associated with the DFT is satisfied once the system reaches steady state.

If the number of averaged spectral lines is greater than or equal to the number of references, and if the phase between the source signals is chosen such that the columns of the reference cross-spectrum matrix, $[X(\omega_i)]$, are independent, then the reference cross-spectra matrix is invertible and FRFs can be calculated.

If the phase between spectral lines is chosen in an optimal manner, then the off-diagonal elements of the reference cross-spectra matrix can be numerically zero for the Daniell spectral estimation method. An example using optimal phase is presented in Equation (3). The optimal phase angles, $[\theta]$, for three frames are defined, and their complex value on a unit circle, $[X]$, is shown where x=w=z, and m=1 in Equation (2). The average of the cross-spectrum of the three frames is then calculated and is equal to the identity matrix.

$$\begin{gathered}
[\theta] = \left[\begin{Bmatrix}0\\0\\0\end{Bmatrix}\begin{Bmatrix}0\\120\\-120\end{Bmatrix}\begin{Bmatrix}0\\-120\\120\end{Bmatrix}\right], [X] = \left[\begin{Bmatrix}1\\1\\1\end{Bmatrix}\begin{Bmatrix}1\\-1/2-i\sqrt{3/4}\\-1/2+i\sqrt{3/4}\end{Bmatrix}\begin{Bmatrix}1\\-1/2+i\sqrt{3/4}\\-1/2-i\sqrt{3/4}\end{Bmatrix}\right]\\
[S_{XX}] = 1/3\left(\begin{bmatrix}1&1&1\\1&1&1\\1&1&1\end{bmatrix} + \begin{bmatrix}1 & -1/2+i\sqrt{3/4} & -1/2-i\sqrt{3/4}\\ -1/2-i\sqrt{3/4} & 1 & 1/2+i\sqrt{3/4}\\ -1/2+i\sqrt{3/4} & -1/2-i\sqrt{3/4} & 1\end{bmatrix}\right.\\
\left.+\begin{bmatrix}1 & -1/2-i\sqrt{3/4} & -1/2+i\sqrt{3/4}\\ -1/2+i\sqrt{3/4} & 1 & -1/2-i\sqrt{3/4}\\ -1/2-i\sqrt{3/4} & -1/2+i\sqrt{3/4} & 1\end{bmatrix}\right) = \begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}
\end{gathered} \tag{3}$$

Example of three-frame, three-source example of optimal phase assignment leading to zero off-diagonals. Phase angles are specified in degrees, and the matrix [X] are its complex values with a magnitude of 1.

In the case of Daniell spectral estimation, the three frames are applied concurrently and are associated with the three frequency bins of the DFT closest to the frequency range of interest. In the case of Bartlett spectral estimation, the three frames are applied sequentially and their average is calculated. An example using two phase combinations is presented in Fig. 1.

If a realizable reference cross-spectrum matrix with non-zero off-diagonals is required for multiple-input environmental testing, it can be generated by pre- and post-multiplying the diagonal matrix by a complex vector and its Hermitian, respectively. This vector is oftentimes the singular vectors of a desired reference cross-spectrum matrix scaled by their associated singular values.

Since the reference cross-spectra with concurrent phasing have been defined, an inverse DFT can be applied to generate a time-history signal that is a summation of all the sine waves with their associated phases at the center of each frequency bin. This time history can be replicated any number of times to generate multi-reference pseudorandom time-history signals of arbitrary length. In this way, multiple frames can be averaged together to reduce variance error due to measurement noise.

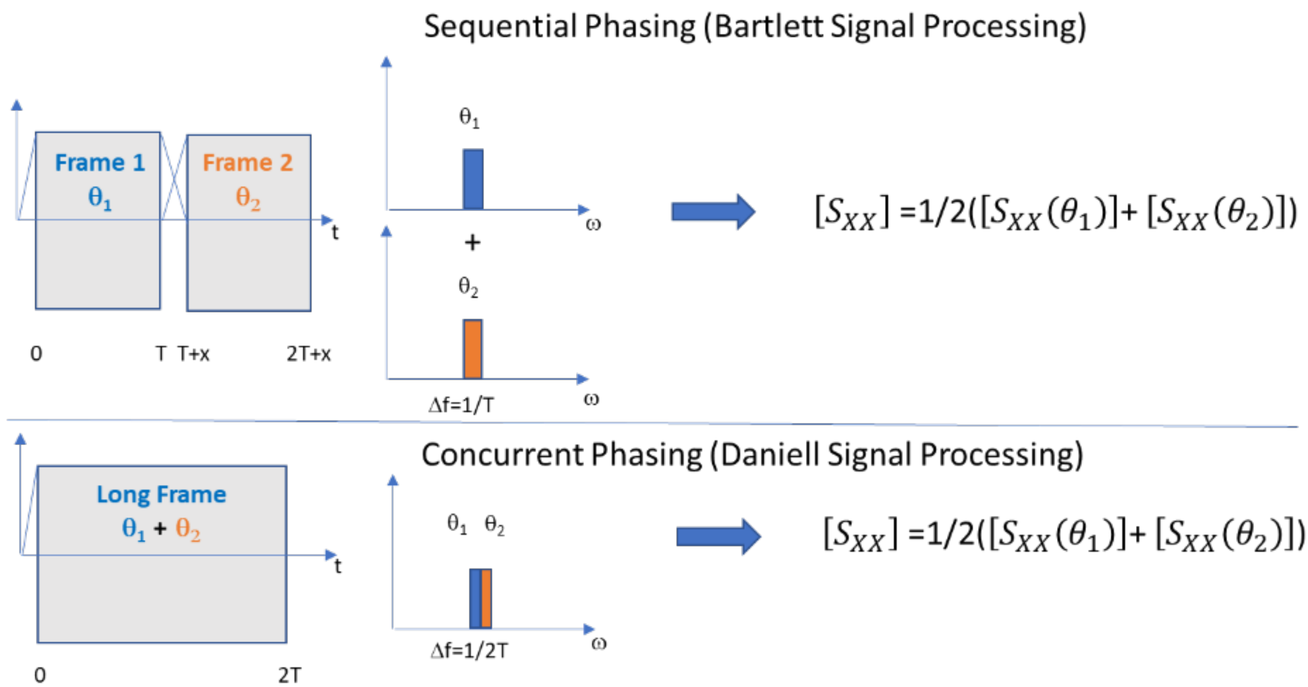

Fig. 1 Example of sequential versus concurrent phase combinations

Furthermore, overlap processing can be used to update the spectral matrices at a faster rate than waiting for a full frame of data to be completed, allowing for the calculation of multiple reference FRFs in the same amount of time as those that are conventionally calculated.

Numerical Examples

This section presents some numerical examples on a lumped parameter model with no noise to demonstrate that the proposed signal definition and signal processing technique can lead to clean reference and response cross-spectral density matrices.

Analysis Model

A seven-degree-of-freedom (DOF) lumped parameter model (Fig. 2) is used to demonstrate the proposed signal generation and signal processing technique. Each mass has a value of 1, each spring has a value of 10,000, and damping was assigned as 1% viscous after modes were solved. Forces are applied at DOFs 1X+, 3X+, and 5X+, and acceleration responses are measured at all seven DOFs. The modal parameters are shown in Table 1.

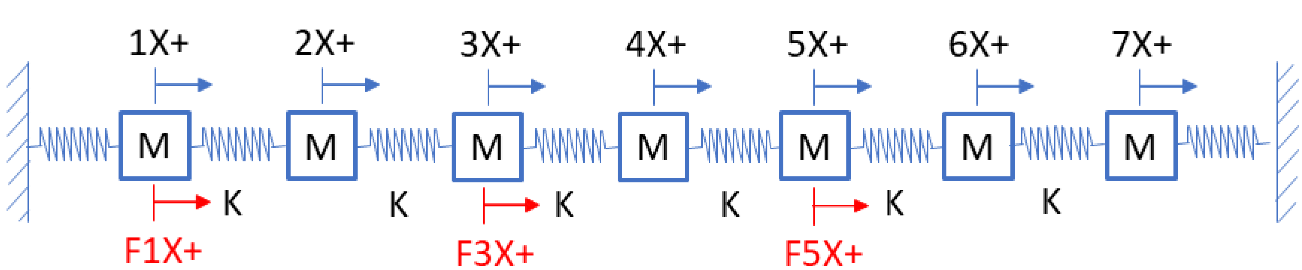

Fig. 2 Seven-DOF lumped parameter model

Table 1 Lumped parameter modal properties

Mode No.	Frequency(Hz)	Damping(% critical)
1	6.21	1
2	12.18	1
3	17.68	1
4	22.51	1
5	26.47	1
6	29.41	1
7	31.21	1

Time-History Signal Generation

The pseudorandom input signals are generated by first calculating the frequency-domain inputs. The frequency-domain inputs $X(\omega)$. The FRF matrix, $H(\omega)$ is used to calculate the outputs in the frequency domain, $Y(\omega)$. Once calculated, both the inputs and outputs are converted to the time domain, $x(t)$ and $y(t)$ and are then replicated as many times as required to obtain the desired time-history length.

The true random input was generated in the time domain and zero-padded at the beginning and end to satisfy the assumptions of the DFT, which was applied to generate the frequency-domain input. The FRF matrix was then used to generate the output frequency-domain results, which were then transformed to the frequency time domain. The original input time-history and output time-history functions were then used for signal processing.

Signal processing techniques are then used to calculate frequency domain functions (Fig. 3) such as cross-spectra, reference cross-spectra, response cross-spectra, and FRFs.

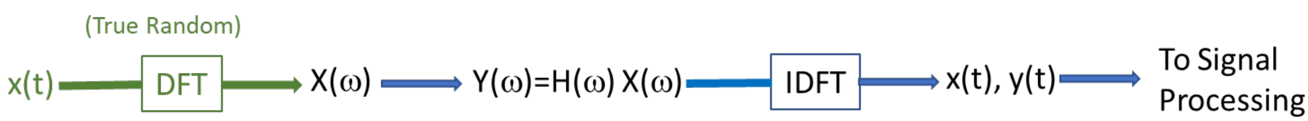

Fig. 3 Response time-history calculation

Case 1: True Random

The first case presented uses a true random excitation as the input. Frequency-domain results using the Welch spectral estimation method [7] are shown in Fig. 4. Ten minutes of simulated test data were processed using a Hanning window and 90% overlap processing to calculate FRF and spectral matrices. The Daniell method (Fig. 5) is applied by zero padding before and after the excitation is activated and turned off to satisfy the assumptions of the DFT that the signal is completely observed. As expected with a true random excitation, the FRF results are clean and there is good separation between the off-diagonal terms of the reference cross-spectra matrix and the diagonal terms. The noise in the reference auto-spectrum and its off-diagonal terms results in a cross-spectrum matrix and response auto-spectrum terms with noise.

Case 2: Multi-Reference Pseudorandom with Random Phase Assignment

The second case presented uses a pseudorandom excitation with a random phase assignment for the input signals at each frequency. For this case, the desired frequency resolution for the processed signals is 0.05 Hz, and the number of frequency bins per analyzed frequency was selected to be 5, which helped minimize the risk of generating reference cross-spectrum matrices with high condition numbers. Therefore, the frequency bin width to generate the signal was 0.01 Hz (0.05 Hz/5 frames), which corresponds to a 100-second frame of simulated test data. The 100-second frame of data was replicated six times to extend the total active time to ten minutes. Frequency-domain results using the Welch spectral estimation method are shown in Fig. 6. Ten minutes of data were processed using a Hanning window and 90% overlap processing to calculate FRF and spectral matrices. Since the input time history repeats every 100 seconds, the Daniell method (Fig. 7) is applied by selecting any 100-second set of data after the transient effects at startup have settled. After settling, the periodic signals satisfy the assumption of the DFT. Overlap processing of 100-second frames is also performed at a 90% overlap rate using the steady-state data up until the excitation source is turned off. It should be noted that because the simulated data in this paper is noise-free, results using one 100-second frame are the same as using many frames. As expected, the FRFs are still

Fig. 4 True random, Welch

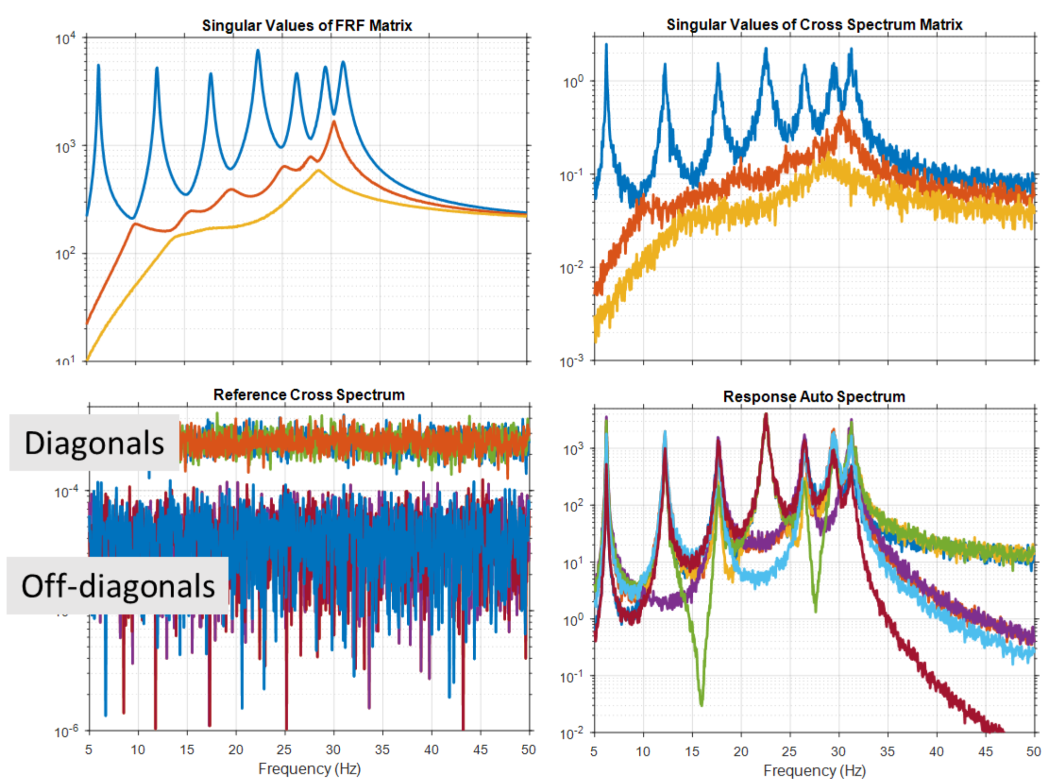

Fig. 5 True random, Daniell

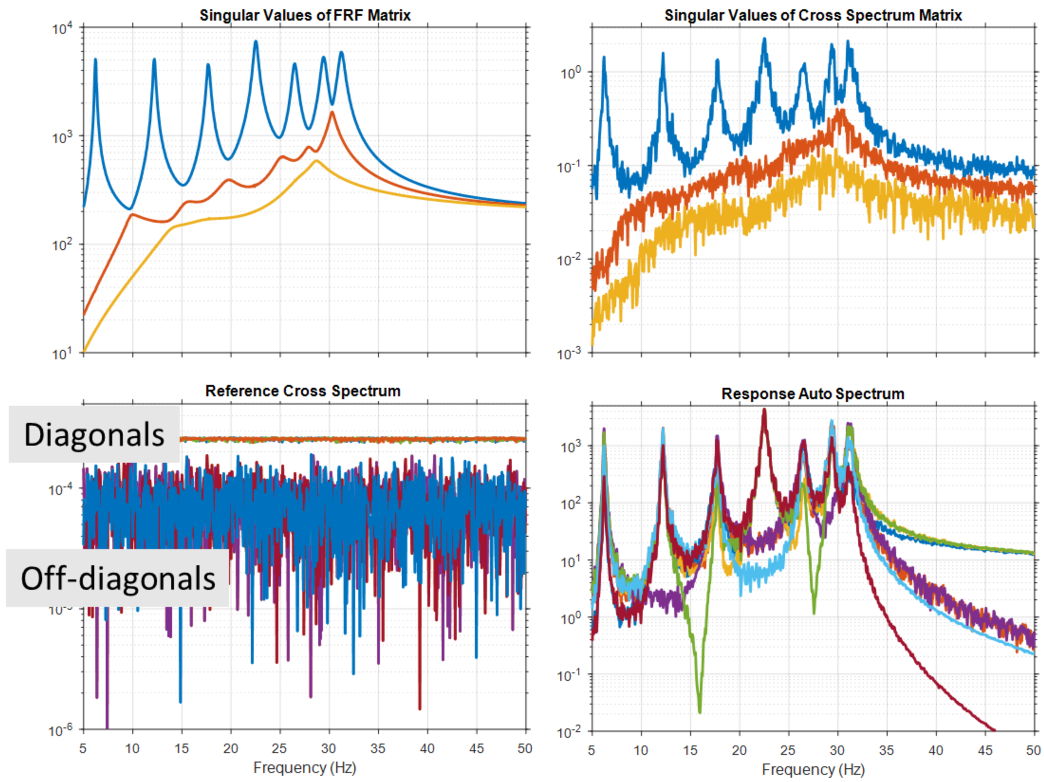

Fig. 6 Multi-reference pseudorandom with random phasing, Welch

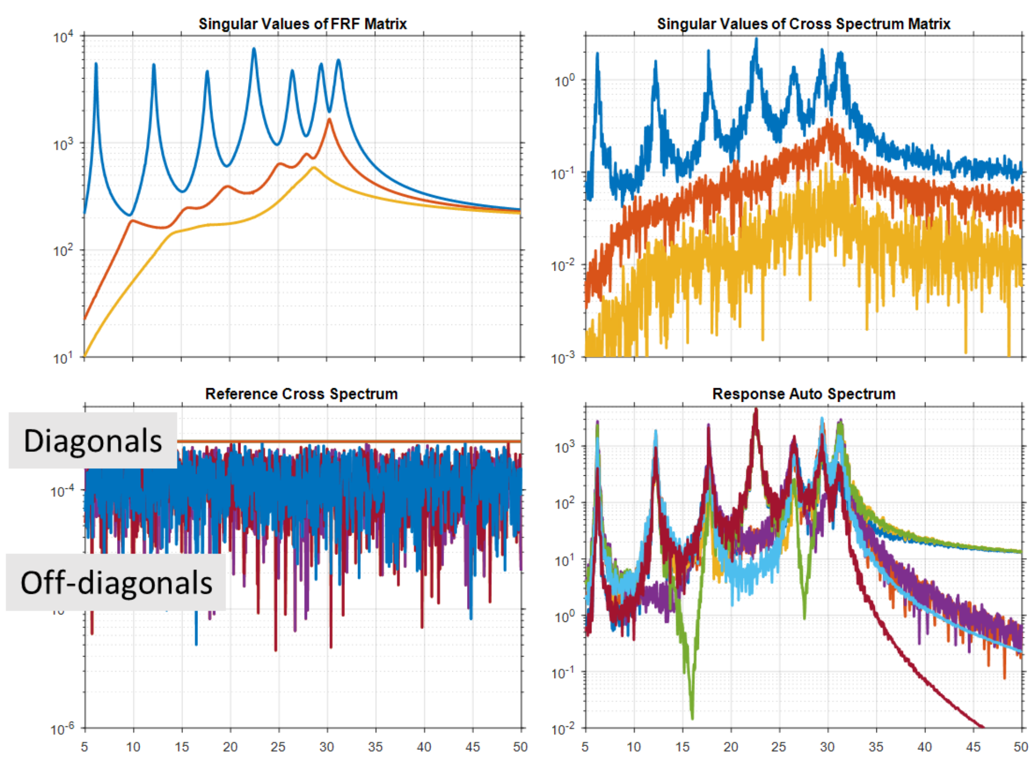

Fig. 7 Multi-reference pseudorandom with random phasing, Daniell

clean, and the diagonals of the reference cross-spectrum matrix are mostly flat; in fact, the diagonals are constant for Daniell spectral estimation. The off-diagonal terms of the reference cross-spectrum matrix for both signal processing cases are higher than the previous case. The “noise” in the off-diagonal terms results in noisy response cross-spectra and auto-spectra data.

Case 3: Multi-Reference Pseudorandom with Optimal Phasing

The third case presented uses a pseudorandom excitation with optimal phase assignment for the input signals at each frequency. For this case, the desired frequency resolution for the processed signals is 0.05 Hz, and the number of frequency bins per analyzed frequency was selected to be 5. Therefore, the frequency bin width to generate the signal was 0.01 Hz (0.05 Hz/5 frames), which corresponds to a 100-second frame of simulated test data. The phase was assigned to each set of five 0.01 Hz frequency bins used to calculate the cross-spectrum of the resultant 0.05 Hz frequency spacing such that the off-diagonal terms of the reference cross-spectrum matrix were minimized. The 100-second frame of data was replicated six times to extend the total active time to ten minutes. Frequency-domain results using the Welch spectral estimation method are shown in Fig. 8. Ten minutes of data were processed using a Hanning window and 90% overlap processing to calculate FRF and spectral matrices. Since the input time history repeats every 100 seconds, the Daniell method (Fig. 9) is applied by selecting any 100-second set of data after the transient effects at startup have settled. After settling, the periodic signals satisfy the assumption of the DFT. Overlap processing of 100-second frames was also performed at a 90% overlap rate using the steady-state data up until the excitation source is turned off. As expected, the FRFs are still clean, the diagonals of the reference cross-spectrum matrix contain less noise, and the off-diagonal terms are lower than Cases 1 and 2. However, for Daniell spectral estimation, the off-diagonal terms are mathematically equal to zero. While both signal processing tech-

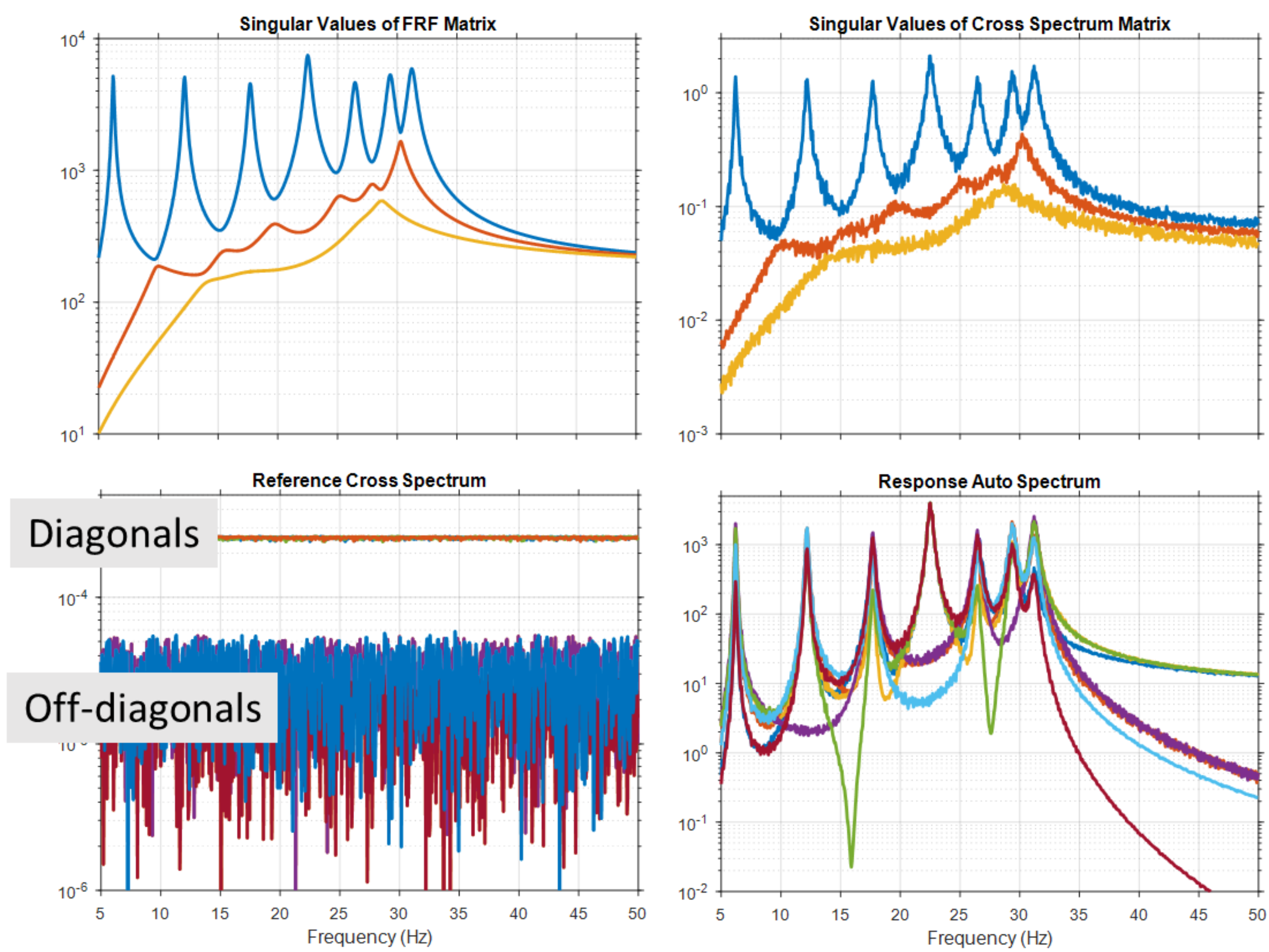

Fig. 8 Multi-reference pseudorandom with optimal phasing, Welch

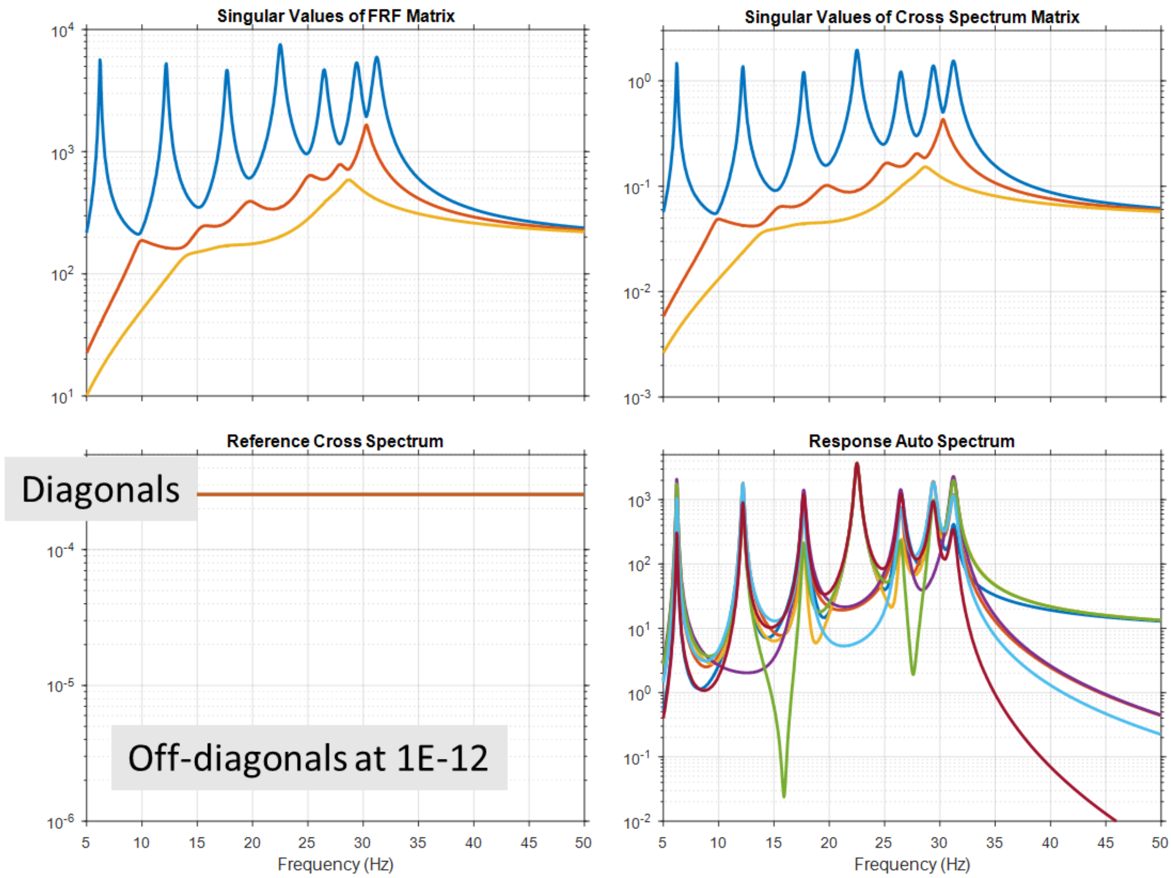

Fig. 9 Multi-reference pseudorandom with optimal phasing, Daniell

niques result in cleaner cross-spectrum and response cross-spectrum functions than Case 1 and 2, the results are noise-free with Daniell spectral estimation.

Summary

This paper has presented a method to define an optimally phased, multi-reference pseudorandom source signal that can be used to generate cleaner spectral matrices. Pseudorandom with concurrent optimal phasing addresses the disadvantages of traditional consecutively phased pseudorandom excitation including fewer skipped transient frames and having the ability to apply a consistent signal over an entire test. The matrices can be further cleaned by using Daniell spectral estimation that matches the optimally phased source signal definition. The ability to generate clean spectral matrices could be useful for more precise definition of multi-input/multi-output environmental control algorithms, or it could be used for operational modal survey test applications where cross-spectral density matrices are used directly to estimate modes, such as acoustic modal applications.

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