Chapter 7
Modelling Homogeneous Populations of Structures using the Statistical Finite Element Method (statFEM)

Brandon J. O'Connell, Keith Worden, and Timothy J. Rogers

Abstract Population-based structural health monitoring (SHM) promises to leverage diagnostic information from multiple systems for a shared benefit. To move towards population-based prognosis, physical models are required. Incorporation of uncertainty across population models will be a key technology to achieve this goal. The Statistical Finite Element Method or statFEM has recently emerged as a new method of combining stochastic finite element (FE) models with noisy observed data in a Bayesian framework, to obtain a posterior over possible solutions. Despite its promising capabilities, case studies exploring the application of this technology, particularly to structural applications, remain limited. Turning to the population-level approach, the current work will demonstrate a population-based application of statFEM to simulated data from multiple structures. In this study, a statFEM model of a simple structure – a clamped-clamped beam – will be demonstrated where a homogeneous population of beams can leverage a shared underlying statistical representation despite having unique variations in properties. Via an implementation of statFEM, it will be shown how one can recover posterior densities over the model solutions, using a shared prior FE, given data from multiple structures.

Keywords Finite element · Population-based SHM · statFEM

Introduction

Population-based approaches to data-driven learning have arguably brought about a new paradigm in the field of structural health monitoring (SHM), with several proposed methods aiming to address the problem of data scarcity. The resulting field of *population-based SHM* (PBSHM) has thus seen considerable growth in the last few years [1, 2]. Despite significant work on advancing data-driven approaches for PBSHM, little research has been conducted on population-based modelling for SHM, and by extension model updating, particularly under uncertainty.

As practitioners increasingly seek new and more informative ways to make decisions, combining physics-based models with measured data under uncertainty [3], population-based modelling via finite element (FE) methods are of interest, particularly where structures are often considered nominally the same *i.e*. a homogeneous population [1]. Of the available approaches to uncertainty quantification in FE modelling (FEM), the *statistical finite element method* (statFEM) [4] has emerged as a new promising solution to this problem and one that provides the necessary Bayesian framework to consider a population-level approach.

Although statFEM has been applied to a selection of simple PDEs in the original and related works [4, 5, 6], the application of statFEM to problems in structural mechanics remain limited. Few examples are present in the literature, notably the work on the Staffordshire bridge [7], and that applying statFEM to buckling of railway sleepers [8]; both of which apply simplified forms of the full statFEM approach. In both cases only the force or the model parameter are modelled as being uncertain, not both simultaneously. However, in the setting of PBSHM one expects to see unknown variability in both the model parameters and the loading conditions. Therefore, consideration of uncertainty from both sources is desired.

In this paper, application of statFEM to the static elasticity problem in structural mechanics is demonstrated and applied to a population of simulated clamped-clamped beams. A shared prior FE model is generated, assuming that the forcing

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and a model parameter are stochastic. This FE prior is then combined with (simulated) observation data to obtain posterior densities over the possible FE solutions. Two different health states are shown, with the observation data for these systems generated using two FE models with known parameters, respectively.

Background

The incorporation of uncertainty in FE and model updating procedures is not a new discipline but has seen continuous and growing attention. Particular areas of interest include computational efficiency [9], Bayesian approaches [10], and the scalability of such methods [5]. One research challenge has been the coalescence of the two fields of stochastic FE and statistical FE; where one aims to unify stochastic models and noisy data in a statistical framework. In stochastic FE, model parameters (*e.g.* material properties, loads) are considered uncertain and treated probabilistically. These uncertainties are then propagated through an FE model to obtain probabilistic outputs [11, 12]. Alternatively, in statistical FE traditional models are combined with noisy data using statistical inference to obtain updated and improved estimates of the system under investigation [13]. Often such an approach is attempted within a Bayesian framework [13, 14].

The *statistical finite element method* or statFEM, formally introduced by Girolami in 2021 *et al*. [4], provides a unifying approach to these challenges under a fully Bayesian framework; allowing synthesis of observed noisy data and stochastic FE models to obtain a posterior density over the possible set of solutions. In the formulation of statFEM, the forward (stochastic FE) problem considers the solution to the PDE as random field (see [11]) and is solved as a probabilistic FE. This is achieved via the introduction of Gaussian processes over the uncertain model parameters and by propagation of the uncertainty through the classic FE problem. Following discretisation of the relevant domain, the prior probability density over the FE solution is obtained using a first-order perturbation (Taylor series approximation). The resulting density has a mean, given by the standard FE solution, and an associated covariance (Eq. (27) in [4]); thus approximating the density as Gaussian. It is noted by Girolami *et al*. that this approximation is only valid for small variances in the model parameters.

Statistical inference is then performed by decomposing the observed data y into three random components (see [15]); a finite element component u, a model mis-specification component d, and a measurement noise component e, to give

$$y = \rho P u + d + e\ , \tag{1}$$

where ρ is a random hyperparameter and P is a projection operator. Given access to the full joint density, the posterior densities over each of the random variables can then be obtained in the usual way, by employing Bayes' rule. In practice, the misspecification and measurement components are considered to be Gaussian processes with some chosen kernel and controlled by a set of learnable hyperparameters.

Although initially shown for Poisson's equation in the original work, statFEM can be adapted to the problem of static deflection because the elasticity PDE is also elliptic by definition [16]. Using the weak form of the general elasticity PDE, one obtains the following discrete system of linear equations,

$$K(\theta)u = f\ , \tag{2}$$

where $K(\theta) \in \mathbb{R}^{n_u, n_u}$ is the stiffness matrix, $\theta \in \mathbb{R}^{n_e}$ is the vector of uncertain model parameters, $u \in \mathbb{R}^{n_u}$ is the vector of nodal solutions, and $f \in \mathbb{R}^{n_u}$ is the uncertain nodal load vector.

Method

Using a custom-built FE package in Python, a 200 node Timoshenko beam element mesh was generated for a rectangular cross-section (prismatic) aluminium beam using the parameters defined in Table 1. A mesh convergence study was used to find the appropriate number of elements to ensure verification against the theoretical result. Using the same base mesh, several alternate health states were simulated by varying the thickness along the beam, equivalent to a scaling on the Young's modulus. The functions of thickness used for each state are given in Table 2. Given that traditional Timoshenko beam elements assume prismatic beams *i.e*. a constant cross-section [16], the mean thickness in each element was calculated using the corresponding values of thickness function at the element nodes.

To incorporate variation in the loading, the amount of force applied to the beam was changed in each scenario from that of the prior. The values of force are also provided in Table 2. In this study, the applied load was distributed evenly between the two nodes either side of the mid-span ($x = 0.25$), replicating a simple point load. In all examples, the beam was subjected to Dirichlet boundary conditions at either end, ensuring a clamped-clamped setup.

Table 1 Geometric and Material Properties

Parameter	Value	Units
Length (l)	0.5	m
Height (h)	0.02	m
Thickness (t)	0.02	m
Cross-Sectional Area (A)	th	m^2
Second Moment of Area (I)	$th^3/12$	m^4
Young's Modulus (E)	70	GPa
Shear modulus (G)	26	GPa
Poisson's Ratio (ν)	0.33	-
Shear Correction Factor (κ)	5/6	-

Table 2 Health scenarios and their associated thickness and applied load

Scenario	Thickness Function $t(x)$ (m)	Applied Load P (N)
1 (Healthy - Prior)	0.020	3500
2 (Damaged)	0.017	3550
3 (Damaged)	$0.002\,(1-x)\cos(2\pi x) + 0.018$	3200

Prior Finite Element

To obtain the prior FE solution, Gaussian processes (GPs) were introduced over the thickness t (*i.e.* the uncertain model parameter) and the applied force f, both modelled by squared-exponential kernels $K_t\left(x, x'\right)$ and $K_f\left(x, x'\right)$, such that

$$t \sim GP\left(\bar{t}, K_t\left(x, x'\right)\right), \tag{3}$$

and

$$f \sim GP\left(\bar{f}, K_f\left(x, x'\right)\right) \tag{4}$$

where (see [17]),

$$K_{\mathrm{SE}}(x, x') = \sigma^2 \exp\left(-\frac{(x - x')^2}{2\ell^2}\right) \tag{5}$$

In line with [4], the prior mean is equivalent to the classical FE solution which can be easily obtained using traditional FE approaches. The mean is computed using the prior mean force $\bar{f}$ and prior mean thickness $\bar{t}$ (Scenario 1 in Table 2), such that

$$\bar{u} = K(\bar{t})^{-1}\bar{f}\,. \tag{6}$$

Using a custom Python implementation of statFEM developed by the authors, the prior covariance over the FE solution was also computed using Eq.(27) from [4]. Given the element stiffness matrix for a Timoshenko beam element [18], the derivative of the element stiffness with respect to the thickness t, could be easily obtained. The hyperparameters of the two GPs were constrained to fixed values, as detailed in Table 3. These parameter values were chosen sensibly, given insight of the problem but could be optimised if required. The calculated prior density over the FE solution is shown in Figure 1.

Table 3 Hyperparameters

Hyperparameter	Value
σ_t	0.04
ℓ_t	100
σ_f	10
ℓ_f	10

In each axis, the mean solution is given by the blue dot-dashed line, and variance by the incremented shaded regions. The top axis in Figure 1 displays the y (transverse) displacement and the corresponding variance, the middle axis displays

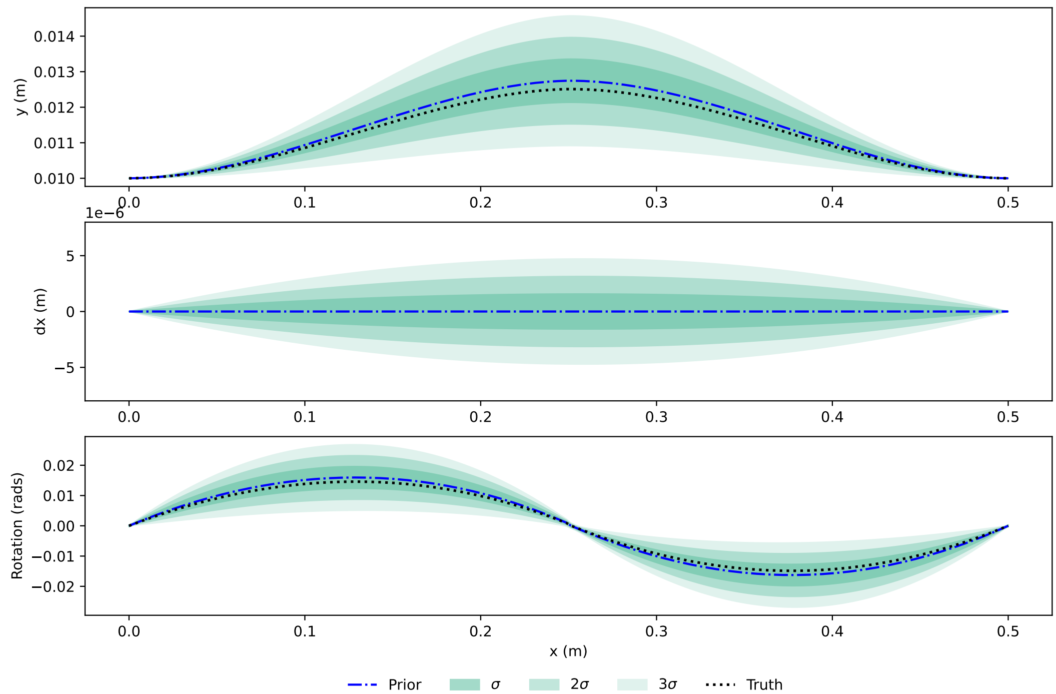

Fig. 1 Prior density over the finite element solution computed using statFEM. Top: transverse displacement. Middle: difference in axial displacement. Bottom: shear rotation angle

the difference in the axial displacement, and finally the bottom axis shows the shear rotation angle. The axial displacement and covariance of beams is difficult to interpret in any case, but is included here as the difference in displacement between the mesh and the prior for completeness.

Results and Discussion

In this section, the results from the application of statFEM using data generated for two health scenarios are shown. Recall that the 'damage' in this study corresponds to a variation in the beam thickness along the length, as defined in Table 2. Ten measurements of the vertical displacement were generated at each of the ten observation locations, spaced evenly along the beam, directly from other FE models constructed with the true parameters. The posterior density over the FE solution was then obtained using statFEM, combining the prior FE with the observed data. The hyperparameters chosen for the discrepancy d and the noise variance e are given in Table 4.

Table 4 Hyperparameters

Hyperparameter	Value
σ_d	10^{-7}
ℓ_d	100
σ_e	10^{-6}

Figures 2 and 3 show the posterior densities over the FE solution for Damage Case One and Damage Case Two, respectively. As evidenced by the posterior, These results demonstrate that statFEM can be successfully used to update a (shared) FE prior density and converge toward the true underlying system, given noisy observations of a system.

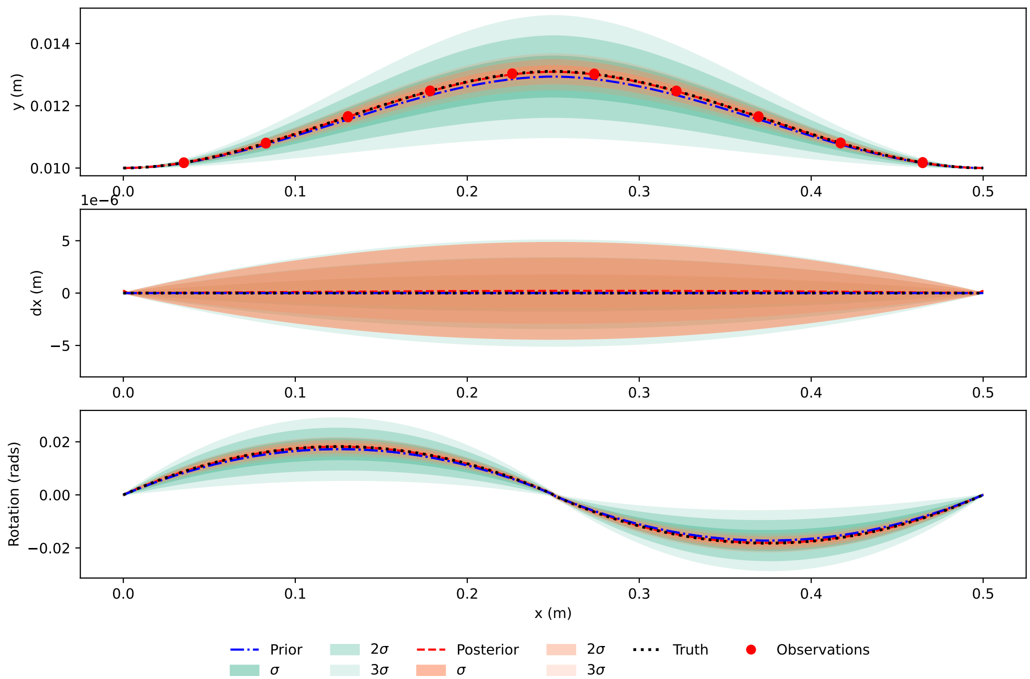

Fig. 2 StatFEM Posterior FE solution for Damage Case 1

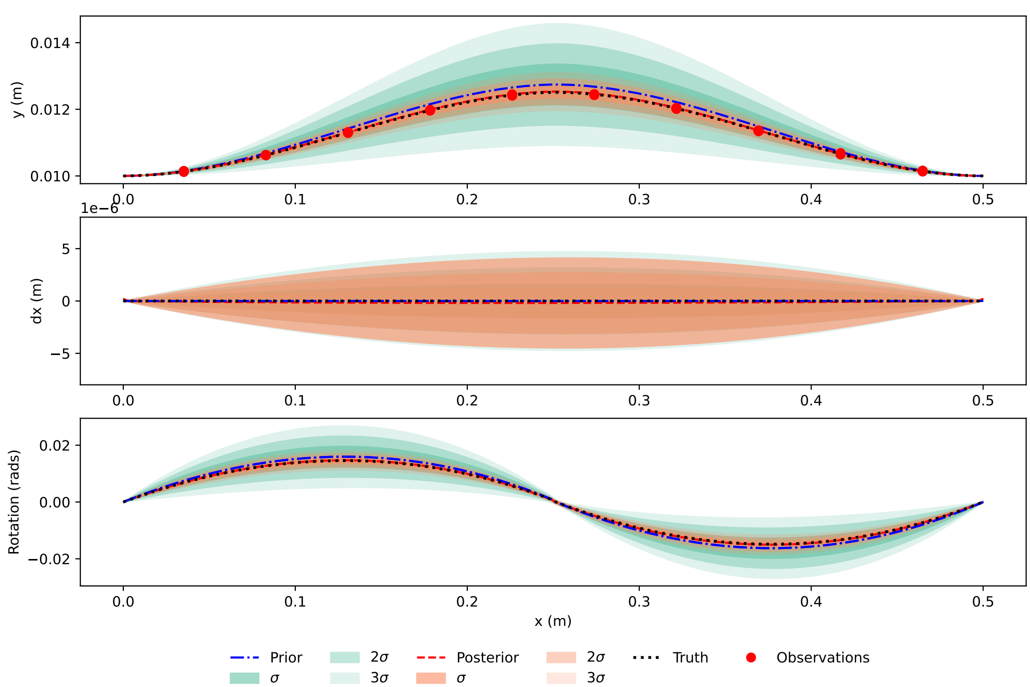

Fig. 3 StatFEM Posterior FE solution for Damage Case 2

Interestingly, the uncertain models arising from statFEM (considering both the prior and posterior densities) can be considered to constitute what is known as a *Form*; in the context of PBSHM [1]. A Form is that which encompasses both the essential nature of an object of interest, *i.e*. the mean FE solution of the cantilever beam, together with a measure of the expected variation from the ideal across a population, *i.e*. the probability density over possible solutions. Note that a Form need not be an object, but could also exist as a feature. In this case, the prior distribution provides an initial Form, based on pre-existing knowledge of the uncertainty in the parameters for a population, while the posterior gives an updated-Form, on observation of the real system in the physical world. Given access to the uncertainty over possible solutions, the prior outputs of statFEM (a Form) could be considered a general representation of a homogeneous population, which could be used to assist diagnostic tasks in PBHSM.

Conclusion

This work has illustrated how the statistical finite-element method (statFEM) can be adapted to the problem of static mechanics and used to model the prior uncertainty over the static deflection of a structure assuming both uncertainty in the force and the stiffness, given uncertainty in a model parameter (*e.g*. the thickness or Young's Modulus). Assuming uncertainty in the thickness and the applied load, the prior density over the finite element (FE) solution was obtained for the forward problem using the statFEM approach. Simulated data from two alternate health states were then used to obtain the posterior densities over the two beams. In each example, it was shown how statFEM accurately recovers a posterior mean corresponding to the true underlying system whilst also providing an updated estimate for the variance.

The statFEM approach to stochastic and statistical FE has the potential to provide a cohesive and unified approach to Bayesian model updating and digital twinning of structural systems under static loads. Future work aims to apply this technology to physical, real-world structures through an experimental campaign designed to validate its applicability to measured data. Moreover, the authors aim to further explore the methods applications to population-based learning and population-based SHM.

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