# Using Nondimensional Analysis to Model, Design, and Test Hydraulic Control Systems

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## Abstract

This paper presents the merits for modelling and designing hydraulic control systems using nondimensional analysis. It is shown in this paper that nondimensional analysis provides physical insight into the most important features of the control system, facilitating model reduction, providing a basis for machine-design sensitivity analysis, and reducing laboratory experiments. In engineering applications, nondimensional analysis is commonly used for fluid-thermal systems that depend on nondimensional groups such as the Reynold Number, the Mach Number, the Nusselt Number, among others. Even in dynamical systems we use the nondimensional damping ratio to describe the time response of a second order system; however, it is the author's opinion that nondimensional analysis presents a powerful technique for studying hydraulic control systems that should be exploited more fully in our industry. In this paper, the author will present three case studies to show the value of using nondimensional analysis for modelling a hydraulic control system, designing hydraulic machinery, and reducing laboratory experiments. By following these examples, it is believed that efficiency will be improved in bringing products to market, and the quality and performance of our products will be advanced.

**Keywords**. Hydraulic control systems, nondimensional analysis

## 1. INTRODUCTION

Most engineering work is done using dimensions, because we ultimately need dimensions to build physical systems. For example, we need to know if the piston diameter should be 25.4 mm or if that is too small or too large. So, in the end, we will always revert to a dimensional value of a parameter when building hydraulic control systems. However, the physical world really doesn't care if we measure our parameters in pounds, newtons, or stones. And in fact, the physics of a problem are often clouded by those dimensions. For that reason, scientific analysis is usually done in nondimensional form. The thesis of this paper states that nondimensional analysis is also useful for engineering applications; and in this paper we illustrate this fact for the modelling, designing, and testing of hydraulic control systems.

This paper is more of a didactic summary rather than a research paper. As such, the references in this document are used to draw examples from the author's past work to show how nondimensional analysis may be used to improve our understanding of hydraulic control systems. The illustrations are germane to modelling the dynamics of the system, designing machine parameters, and testing operating efficiency.

## 2. MODELLING HYDRAULIC CONTROL SYSTEMS

Hydraulic control systems are often used as an example to teach courses on dynamic systems and control where a strong focus is placed on mathematical modelling. Many examples are presented in such a class to show that dynamical systems may be simplified if their fundamental equations are nondimensionalized. This method of modelling applies to hydraulic systems as well. **Figure 1** presents a dynamical system that we will use to illustrate the benefits of modelling hydraulic control systems using nondimensional analysis.

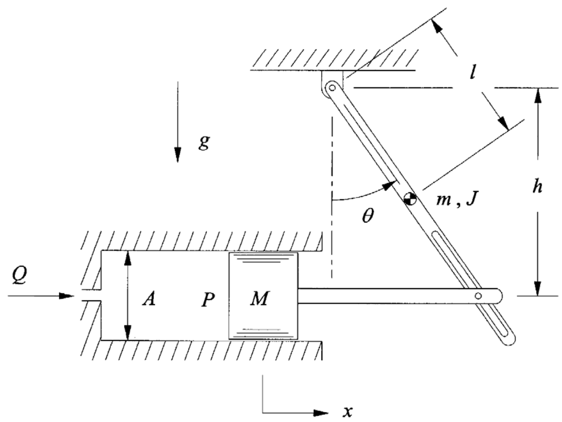


**Figure 1.** Example dynamic system used to illustrate nondimensional modelling [1]

We begin by presenting the model in its dimensional form. Using standard Newtonian mechanics and the conservation of fluid mass, it may be shown that the dynamic equations for this system are given by

$$M\frac{d^2x}{dt^2} = PA - c\frac{dx}{dt} - f \tag{1}$$

$$(J+ml^2)\frac{d^2\theta}{dt^2} = -mgl\sin(\theta) - b\frac{d\theta}{dt} + fh \tag{2}$$

$$\frac{V}{\beta}\frac{dP}{dt} = Q - \mathrm{K}P - \frac{dV}{dt} \quad , \tag{3}$$

In these equations the symbols $c$ and $b$ represent coefficients of viscous damping, K is the coefficient for low Reynolds number leakage, $\beta$ is the fluid bulk modulus, $f$ is the reaction force at the sliding pin-joint, and all other symbols are self-explanatory. The volume of fluid in the piston chamber is given by

$$V = V_o + Ax \tag{4}$$

where $V_o$ is the volume of fluid in the piston chamber when $x = 0$. The nonlinearities of Eq. (1) tend to complicate things. If we assume that $\theta$ doesn't depart too far from zero, a Maclaurin series may we written to show that:

$$\sin(\theta) = \theta = \frac{x}{h} \quad . \tag{5}$$

Substituting this result back into Eq. (1) produces the following simplified form of the dynamic equations

$$M_{eff} \frac{d^2x}{dt^2} + C_{eff} \frac{dx}{dt} + K_{eff}\, x = P\,A \quad \text{and} \quad \frac{V_o}{\beta} \frac{dP}{dt} + \mathrm{K}\, P = Q - A \frac{dx}{dt} \tag{6}$$

where

$$M_{eff} = M + \frac{J + m\,l^2}{h^2} \quad , \quad C_{eff} = c + \frac{b}{h^2} \quad \text{and} \quad K_{eff} = \frac{m\,g\,l}{h^2} \quad . \tag{7}$$

Equation (6) is the simplest, dimensional form of the dynamic equations that we can write. This equation represents a third-order dynamical system that depends upon 12 dimensional parameters. In other words, we can adjust the dynamical characteristics of this system by changing up to 12 things, but physically speaking things are not that complicated. If we nondimensionalize Eq. (6) the dynamics may be shown to depend on only 3 nondimensional groups. To do this we will make the following definitions where symbols with subscripts are steady-state values for each parameter, symbols with carets are nondimensional and of order one, and $\tau$ is a characteristic time constant of the system:

$$x = h\,\hat{x} \quad , \quad P = P_o\,\hat{P} \quad , \quad Q = Q_o\,\hat{Q} \quad , \quad t = \tau\,\hat{t} \tag{8}$$

Substitution of these definitions into Eq. (6) produces the following equations:

$$\begin{aligned} &\frac{M_{eff}\,h}{\tau^2 A\,P_o} \frac{d^2\hat{x}}{d\hat{t}^2} + \frac{C_{eff}\,h}{\tau A\,P_o} \frac{d\hat{x}}{d\hat{t}} + \frac{K_{eff}\,h}{A\,P_o}\,\hat{x} = \hat{P} \\ &\frac{V_o}{\beta} \frac{P_o}{\tau\,Q_o} \frac{d\hat{P}}{d\hat{t}} + \frac{\mathrm{K}\,P_o}{Q_o}\,\hat{P} = \hat{Q} - \frac{A\,h}{\tau\,Q_o} \frac{d\hat{x}}{d\hat{t}} \quad . \end{aligned} \tag{9}$$

From steady-state analysis of this equation, where all time derivatives vanish, it can be shown that

$$Q_o = \mathrm{K}\,P_o \quad \text{and} \quad P_o = \frac{K_{eff}\,h}{A} \quad . \tag{10}$$

Substituting these results back into Eq. (9) produces the following form:

$$
\frac{M_{eff}}{\tau^2 K_{eff}} \frac{d^2 \hat{x}}{d\hat{t}^2} + \frac{C_{eff}}{\tau K_{eff}} \frac{d\hat{x}}{d\hat{t}} + \hat{x} = \hat{P}
$$

$$
\frac{V_o}{\beta} \frac{1}{\tau \, \mathrm{K}} \frac{d\hat{P}}{d\hat{t}} + \hat{P} = \hat{Q} - \frac{A^2}{\tau \, \mathrm{K} \, K_{eff}} \frac{d\hat{x}}{d\hat{t}} \quad . \tag{11}
$$

If we define the characteristic time as

$$
\tau = \frac{A^2}{\mathrm{K} \, K_{eff}} \tag{12}
$$

and substitute this into Eq. (11) the simplest form of the nondimensional equations becomes

$$
\xi_1 \frac{d^2 \hat{x}}{d\hat{t}^2} + \xi_2 \frac{d\hat{x}}{d\hat{t}} + \hat{x} = \hat{P} \quad , \quad \xi_3 \frac{d\hat{P}}{d\hat{t}} + \hat{P} = \hat{Q} - \frac{d\hat{x}}{d\hat{t}} \tag{13}
$$

where the 3 nondimensional coefficients / groups are given by

$$
\xi_1 = \frac{M_{eff} \, K_{eff} \, \mathrm{K}^2}{A^4} \quad , \quad \xi_2 = \frac{C_{eff} \, \mathrm{K}}{A^2} \quad \text{and} \quad \xi_3 = \frac{V_o \, K_{eff}}{\beta \, A^2} \quad . \tag{14}
$$

We have just reduced the dynamical complexity of the governing equations from depending on 12 dimensional parameters, to depending on only 3 nondimensional groups! That is powerful in and of itself, but there is more to be gleaned from these results. Since hydraulic control systems are known for having a high effort-to-inertia ratio, and since leakage coefficients need to be small for high efficiency, it may generally be shown that $\xi_1 \ll 1$ and its associated term in the equations may be ignored without loss of accuracy. Similarly, if the viscous damping and leakage is small compared to the actuator size it can be shown that $\xi_2 \ll 1$ and its associated term may also be neglected. Finally, if the fluid bulk modulus is large then $\xi_3 \ll 1$ and the compressibility term in the pressure equation may also be ignored. In other words, our nondimensional analysis has provided enough insight to show what physically matters in the equations under typical conditions. Whereas we thought we had a complex third-order dynamical system, in reality we have a fairly simple first-order dynamical system described by the following equations:

$$
\frac{d\hat{x}}{d\hat{t}} + \hat{x} = \hat{Q} \quad \text{and} \quad \hat{P} = \hat{x} \quad . \tag{15}
$$

These equations can be solved by hand for a given input $\hat{Q}$. Once Eq. (15) is solved for $\hat{x}$ and $\hat{P}$, these values can be re-dimensionalized using the definitions in Eq. (8). Time will also need to be made dimensional by using the time constant in Eq. (12) and that is very straightforward.

Since Eq. (15) represents a linear, first-order system, we know that the settling time for the transient response is four times the time constant given in Eq. (12). More explicitly, the settling time is given by

$$t_s = 4 \times \frac{A^2}{\mathrm{K}\, K_{eff}} \tag{16}$$

This equation shows that the settling time is dominated by the actuator size. A large actuator will take longer to reach steady state compared to a small actuator.

## 3. DESIGNING HYDRAULIC MACHINERY

When designing hydraulic machinery, one of the more difficult tasks is to decide which parameter to adjust to make performance improvements. Conceivably, adjusting some parameters may have a minimal effect on the desired outcome. Adjusting other parameters may have a large effect. In this section we show how to use nondimensional analysis to identify the most important parameter to adjust when trying to increase the fill speed of an axial piston pump.

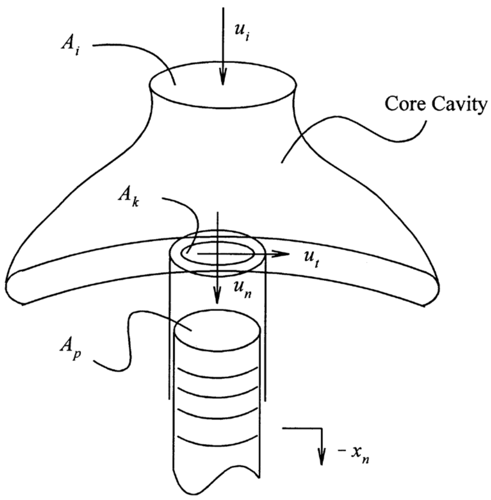


**Figure 2.** A schematic showing a piston chamber filling with fluid [2]

**Figure 2** shows a picture of a single piston within an axial piston pump that is being filled with fluid as the piston withdraws from the piston chamber. The core cavity shows the intake port of the pump. Using Bernoulli principles, it may be shown that the piston will stop filling (and fluid will start to vaporize) when the pump speed exceeds the following limit [2]:

$$\omega < \sqrt{\frac{2\,P_i}{\rho\, r^2 \left\{1+\left[\left(\frac{A_p}{A_k}\right)^2 - \left(\frac{N\,A_p}{\pi\,A_i}\right)^2\right]\tan^2(\alpha)\right\}}} \quad . \tag{17}$$

In this equation $P_i$ is the absolute inlet pressure of the pump, $\rho$ is the fluid density, $r$ is the piston pitch radius, $A_p$ is the cross-sectional area of the piston, $A_k$ is the kidney area on the cylinder block, $A_i$ is the inlet area of the core cavity, $N$ is the total number of pistons in the machine, and $\alpha$ is the swash plate angle of the pump.

To consider the sensitivity of $\omega$ to various design changes, we use a first order Taylor series to study design perturbations about a nominal design condition. Using Eq. (17), we will consider the perturbation of three parameters for illustration. We will perturb the inlet pressure $P_i$, the piston pitch radius $r$, and the kidney area $A_k$. Other parameters could also be studied. The resulting Taylor series is then:

$$\omega = \omega_o + \left.\frac{\partial\omega}{\partial P_i}\right|_o (P_i - P_{i_o}) + \left.\frac{\partial\omega}{\partial r}\right|_o (r - r_o) + \left.\frac{\partial\omega}{\partial A_k}\right|_o (A_k - A_{k_o}) \tag{18}$$

where the subscript "*o*" indicates that the parameters are evaluated at the nominal design conditions. Equation (18) may be rewritten as follows

$$\frac{\omega}{\omega_o} = 1 + \underbrace{\left.\frac{\partial\omega}{\partial P_i}\right|_o \frac{P_{i_o}}{\omega_o}}_{S_{P_i}} \left(\frac{P_i}{P_{i_o}} - 1\right) + \underbrace{\left.\frac{\partial\omega}{\partial r}\right|_o \frac{r_o}{\omega_o}}_{S_r} \left(\frac{r}{r_o} - 1\right) + \underbrace{\left.\frac{\partial\omega}{\partial A_k}\right|_o \frac{A_{k_o}}{\omega_o}}_{S_{Ak}} \left(\frac{A_k}{A_{k_o}} - 1\right) \tag{19}$$

where $S_{P_i}$, $S_r$, and $S_{A_k}$ are nondimensional sensitivity coefficients. Since these coefficients are nondimensional the coefficient with the largest nondimensional magnitude will have the greatest impact on adjusting the fill speed for the pump. It may be shown that these coefficients are given by

$$S_r = \left.\frac{\partial\omega}{\partial r}\right|_o \frac{r_o}{\omega_o} = -1 \quad , \quad S_{P_i} = \left.\frac{\partial\omega}{\partial P_i}\right|_o \frac{P_{i_o}}{\omega_o} = \frac{1}{2} \quad , \quad \text{and}$$
$$S_{A_k} = \left.\frac{\partial\omega}{\partial A_k}\right|_o \frac{A_{k_o}}{\omega_o} = \frac{1}{1+\left(\frac{A_k}{A_p\tan(\alpha)}\right)^2 - \left(\frac{N\,A_k}{\pi\,A_i}\right)^2} = 0.233 \quad . \tag{20}$$

These results are numerically calculated for a 45 cc/rev machine with the following design parameters: $N = 9$, $A_p = 2.28\ \mathrm{cm}^2$, $A_k = 1.52\ \mathrm{cm}^2$, $A_i = 4.56\ \mathrm{cm}^2$, $\alpha = 18\ \mathrm{deg}$, $r = 3.38\ \mathrm{cm}$, $\rho = 850\ \mathrm{kg/m}^3$, and $P_i = 1\ \mathrm{atm}$. As shown in Eq. (20) the most sensitive thing we can adjust to increase the fill speed of the pump is to reduce the piston pitch radius, $r$. Adjustments to this parameter are twice as impactful as changes to the inlet pressure, and four times as impactful to changes in the kidney area. These insights are helpful for design guidance, and are not obvious without the use of nondimensional analysis.

## 4. TESTING HYDRAULIC PUMPS

A common experiment for hydraulic control systems is to measure the efficiency of a hydraulic pump. This is often done by measuring relevant parameters of the machine over a wide range of operating conditions, usually producing hundreds of data points and multiple charts [2]. Using nondimensional analysis, it may be shown that a full understanding of pump efficiency may be obtained by taking only 10 measurements and creating 1 chart.

The *Buckingham Pi Theorem* states that if a physical problem is described by a relevant set of $n$ dimensional parameters which use $m$ independent, fundamental dimensions (e.g., time, length, mass) then a complete set of $n-m$ nondimensional groups may be used to completely describe the problem [3].

Consider our interest in measuring the *overall* efficiency of a hydraulic pump. It turns out that this problem depends on 5 dimensional parameters: discharge pressure $P$, discharge flow rate $Q$, shaft torque $T$, shaft speed $\omega$, and absolute fluid viscosity $\mu$. These parameters are comprised of 3 fundamental dimensions: time, length, and mass. This means that the overall efficiency problem for the pump should be completely described by 2 nondimensional groups. From the relevant set of dimensional parameters, we select these groups to be the overall efficiency and the Hersey number:

$$\eta = \frac{PQ}{T\omega} \quad , \quad H = \frac{\mu\omega}{P} \quad . \tag{21}$$

The Hersey number is often found in fluid mechanics textbooks. Similarly, if we are interested in measuring the *mechanical* efficiency of a hydraulic pump there are 5 dimensional parameters that describe this problem: discharge pressure $P$, volumetric displacement $V_d$, shaft torque $T$, shaft speed $\omega$, and absolute fluid viscosity $\mu$. The 3 fundamental dimensions of this problem are the same as before which means that 2 nondimensional groups may be used to fully describe this problem:

$$\eta_m = \frac{V_d P}{T} \quad , \quad H = \frac{\mu\omega}{P} \quad . \tag{22}$$

Similarly, it may be shown that the *volumetric* efficiency may be described with the following nondimensional groups:

$$\eta_v = \frac{Q}{V_d\,\omega} \quad , \quad H = \frac{\mu\,\omega}{P} \quad . \tag{23}$$

In other words, all efficiency numbers may be plotted on a single chart as they vary with the Hersey number. If a full range of the Hersey number is tested, this chart should describe the efficiency of the pump for the entire operating range of the machine.

The Appendix presents experimental data that was taken while measuring the efficiency of a 145 cc/rev axial piston, swashplate type hydrostatic pump. **Figure 2** shows the efficiency data plotted against the Hersey number. Only 10 data points are used, and the efficiency plot is complete for all operating conditions of the machine. The claim of this nondimensional approach is that only 10 data points and 1 chart is needed for a full description of the pump efficiency across the entire operating range of the machine. This is a significant reduction in engineering effort, and it is also a very useful way to present efficiency data for new product literature.

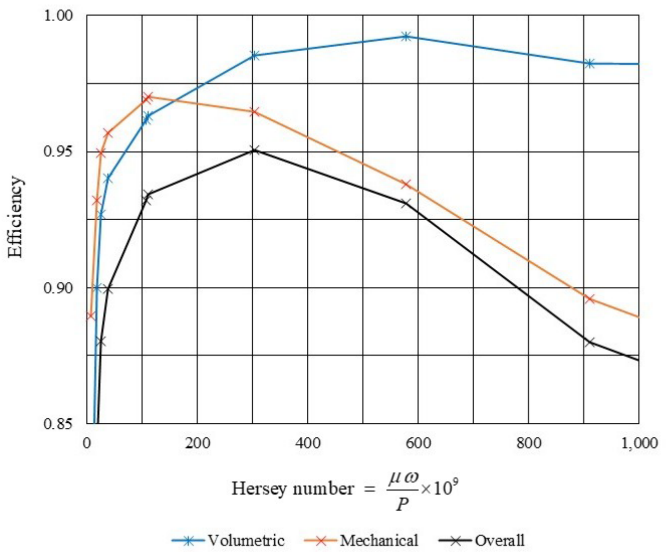


**Figure 2.** A complete map of efficiency for a 145 cc/rev pump

## 5. Conclusion

The following conclusions are supported by the analysis, discussion, and results as presented in this paper:

1. That nondimensional analysis can reduce the complexity of dynamical systems, and that physical insights may be gained which enable the order reduction of the dynamical model.

2. That nondimensional analysis may be used to study the sensitivity of changing machine design parameters, and that this approach elucidates the most important variables to be changed for improved operation.
3. That nondimensional analysis may be used to streamline the experiments that are conducted for the efficiency of hydraulic machinery, and that this approach may be used to reduce experimental effort.

As noted in the introduction of this paper, it is important to always re-dimensionalize the analysis in order to build an actual hydraulic control system. This re-dimensionalization is straightforward using the definitions that were made to conduct the nondimensional analysis.

## 6. APPENDIX

The following table includes the test data that was used to generate **Fig. 2**:

| # | $H \times 10^9$ | $T$ [Nm] | $\omega$ [rpm] | $P$ [MPa] | $Q$ [lpm] | Temp [C] | $\mu$ [cP] | $\eta_v$ | $\eta_m$ | $\eta$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | 7 | 911 | 500 | 34.64 | 57.65 | 112.43 | 4.66 | 0.784 | 0.889 | 0.697 |
| 2 | 19 | 863 | 1,525 | 35.41 | 198.30 | 110.97 | 4.92 | 0.900 | 0.932 | 0.839 |
| 3 | 26 | 749 | 998 | 30.22 | 132.18 | 107.30 | 5.64 | 0.927 | 0.950 | 0.880 |
| 4 | 38 | 735 | 1,999 | 30.06 | 272.87 | 109.03 | 5.29 | 0.940 | 0.957 | 0.900 |
| 5 | 108 | 480 | 1,001 | 20.11 | 141.39 | 63.49 | 28.57 | 0.962 | 0.969 | 0.932 |
| 6 | 112 | 243 | 500 | 9.90 | 70.85 | 53.65 | 41.14 | 0.963 | 0.970 | 0.934 |
| 7 | 305 | 133 | 999 | 5.09 | 144.32 | 51.64 | 44.31 | 0.985 | 0.965 | 0.951 |
| 8 | 578 | 244 | 1,482 | 9.79 | 216.24 | 56.90 | 36.47 | 0.992 | 0.938 | 0.931 |
| 9 | 911 | 147 | 2,178 | 5.11 | 319.64 | 52.88 | 42.32 | 0.982 | 0.896 | 0.880 |
| 10 | 1,345 | 143 | 1,985 | 5.02 | 291.37 | 51.94 | 43.81 | 0.982 | 0.863 | 0.847 |

## 7. REFERENCES

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