Homotopy analysis method (HAM) for solving 3D Heat equations

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ABSTRACT

In this research paper, we present a classical analytical method known as homotopy analysis method for solving three- dimensional heat equations arising in several applications of sciences and engineering. Some numerical experiments have been presented to illustrate the simplicity and accuracy of the presented technique.

KEYWORDS: Homotopy analysis method, Three- dimensional heat equations, Numerical examples.

INTRODUCTION

Three- dimensional partial differential equation has several applications in different branches of sciences and engineering. Numerous numerical techniques have been developed for solving such equations like wavelets method, Adomian decomposition method, Homotopy perturbation method, Finite difference method and many more. In this research, a classical technique has been presented for solving three- dimensional heat equation. In literature, such technique has proven be an efficient technique for solving various mathematical models. In [1], basic concept of homotopy analysis method has been discussed. To find the solutions of nonlinear problems, homotopy analysis method has been presented in [2]. A comparison study of homotopy analysis method and homotopy perturbation method has been explained in [3]. For the solutions of generalized second grade fluid, homotopy based technique has been discussed in [4]. Homotopy analysis method has been presented in [5] for the solutions of boundary layer flow towards stretching sheet. In [6], quadratic Riccati differential equation has been solved with the aid of homotopy analysis method. For the solutions of nonlinear equations produced during heat transfer, homotopy analysis method has been discussed in [7]. In [8], homotopy analysis method has been presented to find the solutions of generalized Hirota- Satsuma coupled with KdV equation. Analytical solutions of Laplace equation with two types of boundary conditions such as Dirichlet and Neumann boundary conditions have been presented in [9]. Analytical solutions of BBMB equations have been discussed with the aid of homotopy analysis method in [10]. Exact solutions of Emden-Fowler type equations have been discussed in [11] by using homotopy analysis method. In [12], homotopy analysis method has been presented to find the solutions of Cauchy reaction-diffusion problems arising in various applications of sciences and engineering.

In this study, we have discussed the classical technique, known as Homotopy analysis method (HAM), which has numerous applications in both science and engineering, for the analytical solutions of three-dimensional heat equations.

The most general form of three- dimensional heat equation is:

$$\eta_t = c^2(\eta_{xx} + \eta_{yy} + \eta_{zz}) - \rho\eta, \quad a < \{x, y, z\} < b, \ t > 0 \tag{1}$$

with initial conditions

$$\eta(x, y, z, 0) = \alpha(x, y, z), \qquad \eta_t(x, y, z, 0) = \beta(x, y, z)$$

where α and β are functions of x, y and z.

BASIC CONCEPTS OF HOMOTOPY ANALYSIS METHOD

Consider the differential equation

$$S\{\eta(\xi)\} = 0, \tag{2}$$

where S represents a nonlinear operator, ξ is an unknown variable and $\eta(\xi)$ is the unknown function. For ease of understanding, disregard all initial and boundary conditions. The zero-order deformation equation as discussed in Liao (2003) is given by:

$$(1 - p)L[\eta(\xi; p) - \eta_0(\xi)] = p\hbar H(\xi)S[\eta(\xi; p)], \tag{3}$$

where the symbol $p \in [0, 1]$ indicates the embedding parameter and $\hbar \neq 0$ represents a supplementary parameter. The function$H(\xi) \neq 0$ denotes a non-zero auxiliary function and L denotes an auxiliary linear operator. The initial guess is taken as $\eta_0(\xi)$ and $\eta(\xi; p)$ represents an unidentified function.
when $p = 0$ and $p = 1$, then the values of $\varphi$are considered as: $\eta(\xi; 0) = \eta_0(\xi)$ and $\eta(\xi; 1) = \eta(\xi)$. The solution $\eta(\xi; p)$ may varies from initial guess $\eta_0(\xi)$ to the unknown function $\eta(\xi)$, when the value of p increases from 0 to1.

Using Taylor's series to expand the function $\varphi(\xi; p)$ as:

$$\eta(\xi; p) = \eta_0(\xi) + \sum_{m=1}^{\infty} \eta_m(\xi)p^m, \tag{4}$$

where

$$\eta_m(\xi) = \frac{1}{m!} \frac{\partial^m \eta(\xi; p)}{\partial p^m}\bigg|_{p=0}$$

The series (4) converges when $p = 1$, then we have

$$\eta(x,t) = \eta_0(x,t) + \sum_{m=1}^{\infty} \eta_m(x,t) \tag{5}$$

When $\hbar = 1$ and $H(\tau') = 1$, the equation (3) becomes

$$(1-p)L[\eta(\xi;p) - \eta_0(\xi)] = pS[\eta(\xi;p)] \tag{6}$$

This relation is mostly used in Homotopy analysis method. Establish the vector

$$\overrightarrow{\eta_n} = \{\eta_0(\xi), \eta_1(\xi), \eta_2(\xi), \dots, \eta_n(\xi)\}$$

The m^th order deformation equation is obtained by differentiating (6), m times with respect to the parameter p, setting $p = 0$, and at the last dividing the result by m!

$$L[\eta_m(\xi) - \chi_m \eta_{m-1}(\xi)] = \hbar H(\xi) R_m(\vec{\eta}_{m-1})$$

where

$$R_m(\vec{\eta}_{m-1}) = \frac{1}{(m-1)!} \frac{\partial^{m-1}\eta(\xi;p)}{\partial p^{m-1}}$$

and

$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & m > 1 \end{cases}$$

Test Examples:

Examples have been provided in this section to show how straightforward and precise the suggested method is.

Example 1: Take a look at the following 3D heat equation

$$\eta_t = \frac{1}{3\pi^2}\{\eta_{\lambda\lambda} + \eta_{\mu\mu} + \eta_{\nu\nu}\},$$

with initial conditions

$$\eta(\lambda,\mu,\nu,0) = \sin\pi\lambda . \sin\pi\mu . \sin\pi\nu,$$

$$\eta_t(\lambda,\mu,\nu,0) = -\sin\pi\lambda . \sin\pi\mu . \sin\pi\nu$$

The exact solution is

$$\eta(\lambda,\mu,\nu,t) = e^{-t} . \sin\pi\lambda . \sin\pi\mu . \sin\pi\nu$$

Choose the linear operator using the Homotopy analysis technique as follow:

$$J[\eta(\lambda,\mu,\nu',t;p)] = \frac{\partial\eta}{\partial t}(\lambda,\mu,\nu,t;p)$$

The linear operator J^{-1} is.

$$J^{-1}(.) = \int_0^t (.)dt$$

Define the term "nonlinear operator" as follow:

$$S[\eta(\lambda,\mu,\nu,t;p)] = \frac{\partial \eta}{\partial t} -{}' \frac{1}{3\pi^2}\left(\frac{\partial^2 \eta}{\partial \lambda^2} + \frac{\partial^2 \eta}{\partial \mu^2} + \frac{\partial^2 \eta}{\partial \nu^2}\right)$$

Create the equation for zeroth-order deformation as

$$(1-p)J[\eta(\lambda,\mu,\nu,t;p) - \eta_0(\lambda,\mu,\nu,t)] = p\hbar\, H(\lambda,\mu,\nu,t)S[\eta(\lambda,\mu,\nu,t;p)]$$

For $p = 0$ and $p = 1$

$$\eta(\lambda,\mu,\nu,t;0) = \eta_0(\lambda,\mu,\nu,t) \text{ and } \eta(\lambda,\mu,\nu,t;1) = \eta(\lambda,\mu,\nu,t)$$

As a result, we arrive at the mth order deformation equations.

$$L[\eta_m(\lambda,\mu,\nu,t) - \chi_m\eta_{m-1}(\lambda,\mu,\nu,t)] = \hbar H(\lambda,\mu,\nu,t)R_m(\vec{\eta}_{m-1}), \quad m \geq 1$$

$$\eta_m(\lambda,\mu,\nu,0) = 0, \qquad (\eta_m)_t(\lambda,\mu,\nu,0) = 0$$

where

$$R_m(\vec{\eta}_{m-1}) = \frac{\partial \eta_{m-1}}{\partial t} - \frac{1}{3\pi^2}\left(\frac{\partial^2 \eta_{m-1}}{\partial \lambda^2} + \frac{\partial^2 \eta_{m-1}}{\partial \mu^2} + \frac{\partial^2 \eta_{m-1}}{\partial \nu^2}\right)$$

The mth order deformation equation has now found its solution, which is

$$\eta_m(\lambda,\mu,\nu,t) = \chi_m\eta_{m-1}(\lambda,\mu,\nu,t) + \hbar H(\lambda,\mu,\nu,t)L^{-1}[R_m(\vec{\eta}_{m-1})], \quad m \geq 1 \qquad (7)$$

We begin by estimating something roughly. $\eta'_0(\lambda,\mu,\nu,t) = \sin \pi x' . \sin \pi y' . \sin \pi z'$, by using the iterative formula (7). Letting $\hbar = -1,\ H(\lambda,\mu,\nu,t) = 1$, Direct access to the other component is possible.

$$\eta_1(\lambda,\mu,\nu,t) = -t . \sin \pi\lambda . \sin \pi\mu . \sin \pi\nu$$

$$\eta_2(\lambda,\mu,\nu,t) = \frac{t^2}{2!} . \sin \pi\lambda . \sin \pi\mu . \sin \pi\nu$$

$$\eta_3(\lambda,\mu,\nu,t) = -\frac{t^3}{3!} . \sin \pi\lambda . \sin \pi\mu . \sin \pi\nu$$

As a result, the parts of $\eta(\lambda,\mu,\nu,t)$ are written as follows:

$$\eta(\lambda,\mu,\nu,t)=\eta_0(\lambda,\mu,\nu,t)+\sum_{m=1}^{\infty}\eta_m(\lambda,\mu,\nu,t)$$

$$\eta(\lambda,\mu,\nu,t)=\sin\pi\lambda.\sin\pi\mu\,.\sin\pi\nu\,.\left(1-t+\frac{t^2}{2!}-\frac{t^3}{3!}+\cdots\right)$$

$$=e^{-t}.\sin\pi\lambda.\sin\pi\mu\,.\sin\pi\nu$$

which is exact solution.

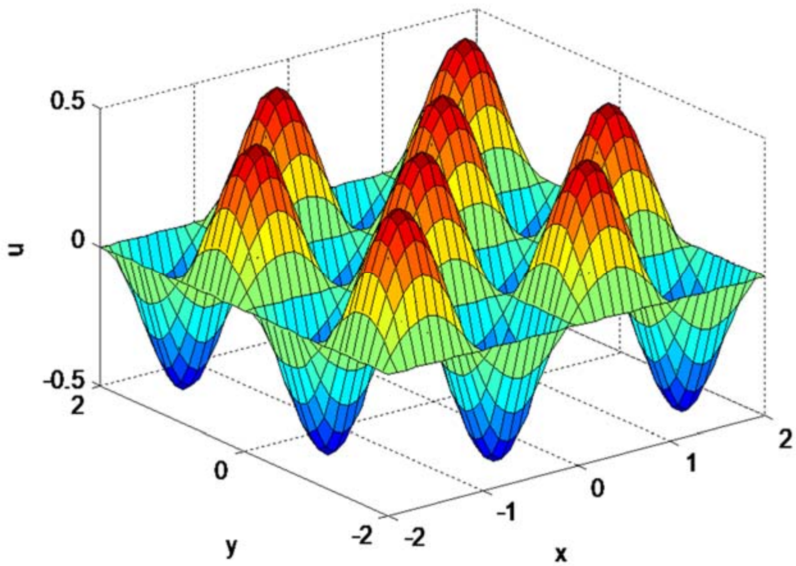

Figure 1

Figure 1 shows the physical behavior of solutions of Example 1 for $t=0.001$ and $\nu=1/6$ and $2\leq(\lambda,\mu)\leq 2$.

Example 2: Consider the following 3D heat equation.

$$\frac{\partial\eta}{\partial t}=\frac{\partial^2\eta}{\partial\lambda^2}+\frac{\partial^2\eta}{\partial\mu^2}+\frac{\partial^2\eta}{\partial\nu^2}-\eta$$

with initial conditions

$$\eta(\lambda,\mu,\nu,0)=\sin 2\pi\lambda.\sin 2\pi\mu\,.\sin 2\pi\nu$$

and

$$\eta_t(\lambda,\mu,\nu,0)=-13\pi^2.\sin 2\pi\lambda.\sin 2\pi\mu\,.\sin 2\pi\nu$$

The exact solution is

$$\eta(\lambda,\mu,\nu,t) = e^{-13\pi^2 t}.\sin 2\pi\lambda.\sin 2\pi\mu\,.\sin 2\pi\nu$$

The linear operator should be chosen using the Homotopy analysis technique.

$$J[\eta(\lambda,\mu,\nu,t;p)] = \frac{\partial\eta}{\partial t}(\lambda,\mu,\nu,t;p)$$

The equation for the linear operator J^{-1} is.

$$J^{-1}(.) = \int_0^t (.)dt$$

Explain a nonlinear operator.

$$S[\eta(\lambda,\mu,\nu,t;p)] = \frac{\partial\eta}{\partial t} - \frac{\partial^2\eta}{\partial\lambda^2} - \frac{\partial^2\eta}{\partial\mu^2} - \frac{\partial^2\eta}{\partial\nu^2} + \eta$$

The zeroth-order deformation equation should be built as.

$$(1-p)J[\eta(\lambda,\mu,\nu,t;p) - \eta_0(\lambda,\mu,\nu,t;p)] = p\hbar\, H(\lambda,\mu,\nu,t)S[\eta(\lambda,\mu,\nu,t;p)]$$

For $p = 0$ and $p = 1$

$$\eta(\lambda,\mu,\nu,t;0) = \eta_0(\lambda,\mu,\nu,t) \text{ and } \eta(\lambda,\mu,\nu,t;1) = \eta(\lambda,\mu,\nu,t)$$

As a result, we have the m^th order deformation equations.

$$J[\eta_m(\lambda,\mu,\nu,t) - \chi_m\eta_{m-1}(\lambda,\mu,\nu,t)] = \hbar H(\lambda,\mu,\nu,t)R_m(\vec{\eta}_{m-1}), \qquad m \geq 1$$

$$\eta(\lambda,\mu,\nu,0) = 0,$$

and

$$(\eta_m)_t(\lambda,\mu,\nu,0) = 0$$

where

$$R_m(\vec{\eta}_{m-1}) = \frac{\partial\eta_{m-1}}{\partial t} - \frac{\partial^2\eta_{m-1}}{\partial\lambda^2} - \frac{\partial^2\eta_{m-1}}{\partial\mu^2} - \frac{\partial^2\eta_{m-1}}{\partial\nu^2} + \eta_{m-1}$$

The m^th order deformation equation has now found its solution.

$$\eta(\lambda,\mu,\nu,t) = \chi_m\eta_{m-1}(\lambda,\mu,\nu,t) + \hbar H(\lambda,\mu,\nu,t)\, L^{-1}[R_m(\vec{\eta}_{m-1})], \quad m \geq 1 \qquad (8)$$

We start with an underlying guess $\eta_0(\lambda,\mu,\nu,t) = \sin 2\pi\lambda.\sin 2\pi\mu\,.\sin 2\pi\nu$, by using the iterative formula (8). Letting $\hbar = -1$, $H(\lambda,\mu,\nu,t) = 1$, we can acquire the other part straightforwardly.

$$\eta_1(\lambda,\mu,\nu,t) = -13\pi^2 t.\sin 2\pi\lambda.\sin 2\pi\mu\,.\sin 2\pi\nu$$

$$\eta_2(\lambda,\mu,\nu,t) = \frac{(13\pi^2 t)^2}{2!}.\sin 2\pi\lambda.\sin 2\pi\mu\,.\sin 2\pi\nu$$

$$\eta_3(\lambda,\mu,\nu,t) = -\frac{(13\pi^2 t)^3}{3!}.\sin 2\pi\lambda.\sin 2\pi\mu\,.\sin 2\pi\nu$$

$$\vdots$$

In this manner, the parts which comprise $\eta(\lambda,\mu,\nu,t)$ are composed as:

$$\eta(\lambda,\mu,\nu,t) = \eta_0(\lambda,\mu,\nu,t) + \sum_{m=1}^{\infty}\eta_m(\lambda,\mu,\nu,t)$$

$$\eta(\lambda,\mu,\nu,t) = \sin 2\pi\lambda.\sin 2\pi\mu\,.\sin 2\pi\nu\,.\left(1 - 13\pi^2 t + \frac{(13\pi^2 t)^2}{2!} - \frac{(13\pi^2 t)^3}{3!} + \cdots\right)$$
$$= e^{-13\pi^2 t}.\sin 2\pi\lambda.\sin 2\pi\mu\,.\sin 2\pi\nu$$

which is exact solution.

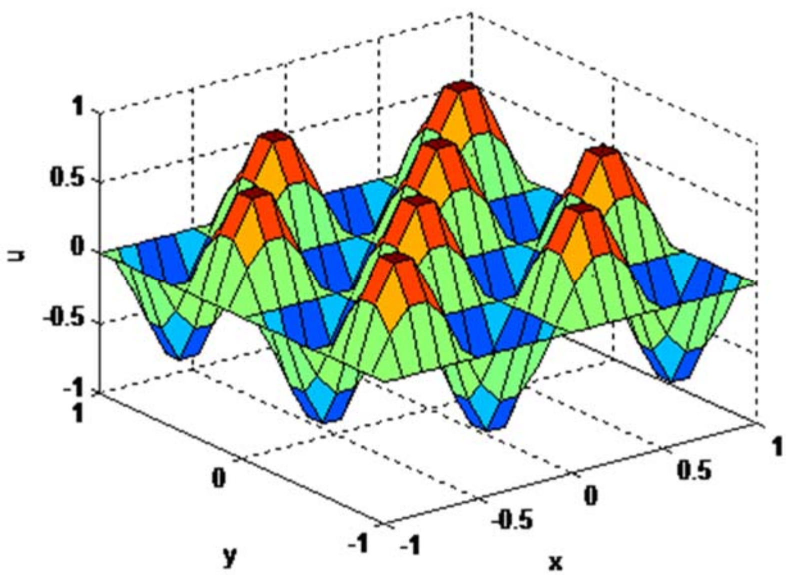

Figure 2

Figure 2 shows the physical behavior of solutions of Example 2 for $t = 0.001$ and $\nu = 1/6$ and $-1 \le (\lambda,\mu) \le 1$.

CONCLUSION

In the perspective on the above delineated models, it is presumed that Homotopy examination strategy (HAM) is extremely strong procedure for tackling three layered heat conditions, which are emerging in numerous utilizations of sciences and designing. For future extension, this strategy will be relevant for various straight and nonlinear incomplete differential conditions as well as three layered fragmentary fractional differential conditions.

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