Adaptive Maintenance Optimization Using Initial Reliability Estimates

Khalid Aboura¹ and Johnson I. Agbinya²

¹College of Business Administration, University of Dammam, Saudi Arabia; e-mail: kaboura@ud.edu.sa
²Department of Electronic Engineering, La Trobe University, Kingsbury Drive, Victoria 3083, Australia; e-mail: j.agbinya@latrobe.edu.au

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Abstract

This paper presents a procedure for determining optimal times for the replacement of a large number of identical items operating under similar conditions. A collective maintenance policy is considered due to the prohibitive cost of individual replacement upon failure. Replacements of all items occur at prescribed points in time, assuming that a reasonable proportion of failed items are tolerated. The first replacement time is chosen using initial reliability estimates. Successive replacement times are determined as failure and survival information is gained. The procedure is developed for the data collection scenarios of complete and interval censored data.

Keywords: Maintenance optimization, block replacement, statistical inference.

1 Introduction

We present an adaptive maintenance optimization solution for the replacement of a large number of identical items operating independently under similar conditions. An example would be the replacement of light bulbs in city streets and large department stores. In a typical situation, individual re-
placement upon failure is expensive and planned replacement of all items is preferred. A minimization of maintenance costs is essential given the sizes of the structures. Upon the introduction of new equipment or at the start of a study, reliability estimates are often available only in the form of vendor information or informed judgment from maintenance operators. As failure and survival data are collected, a better assessment of the life length characteristics of the items becomes possible, allowing a more effective replacement procedure. Adaptive maintenance strategies apply well in this situation. Mazzuchi and Soyer [13] use an updating scheme in a decision theoretic set up to allow the maintenance policy to adapt to failure information. Earlier maintenance optimization algorithms were developed based on the assumption that the lifetime characteristics of the items are specified at the start of the operations. Zio and Compare [21] give a good perspective on the three main aspects of corrective, preventive and dynamic maintenance. For a survey of maintenance models when the parameters of the failure distribution are unknown but constant see Valdez-Florez and Feldman [18] and Sarkar et al. [14]. When the parameters of the failure distribution are considered random variables, a Bayesian parametric analysis often ensues [20]. In such case, a growing number of models present exceptions to the static nature of the early policies. Some of these work include Fox [9], Bassin [4], Bather [5], Frees and Ruppert [10], Aras et al. [3], Benmerzouga [6], Mazzuchi and Soyer [13] and subsequently Sheu et al. [15, 16], Dayanik and Gurler [7], Juang and Anderson [11], and Flage et al. [8]. The maintenance scenario considered in this paper departs from the traditional block and age replacement scenario. Here, an item under consideration is assumed to remain failed upon failure until the next planned replacement time, when all items are replaced. The manual cost of inspection and replacement of items prohibits an individual maintenance. In addition, we assume that the failure of a reasonable proportion of the items does not bring to stop the whole structure and therefore allows a planned strategy for the simultaneous replacement of all items. In the following, the Weibull distribution is assumed to be the lifetime model. A maintenance optimization solution is outlined and simulated examples are used to illustrate the methodology.

2 The Maintenance Scenario

We consider a structure of \( M \) identical items operating independently of each other under similar conditions. At prescribed points in time \( T_1, T_2, \ldots, \) etc., all items are replaced by new ones. An item that fails before the next replace-
ment time remains failed. We let $T_0$ be time 0. As failures accumulate between the replacement times, two types of data collection are possible: (1) the exact failure times are recorded (complete data) and (2) the number of failures per time interval is recorded (interval censored data). We treat both cases and consider only the case of the numbers of failures between replacement times in case (2). That is the interval censored data consist of the number of failures in $[T_{i-1}, T_i)$, $i = 1, 2, \ldots$. The more general case of interval censored data involves inspection points in $[T_{i-1}, T_i)$ where the numbers of failures are recorded between the inspection points. The extension to the inspection case is straightforward given that the inspection times are fixed. A further extension would be to consider the inspection times as decision variables in the setting of an optimal maintenance strategy.

3 The Lifetime Distribution

The lifetime of an item, say $T$, is assumed to have a Weibull probability distribution with reliability function $R(t/\lambda, \beta) = e^{-\lambda t^\beta}$. At time $T_0$, the prior distribution of the parameters $(\lambda, \beta)$ is assessed using available information. Aboura [1] introduces an approach for the construction of a prior distribution for $(\lambda, \beta)$ when initial reliability estimates are available. We use the methodology of Aboura [1] in constructing a prior distribution.

3.1 The Prior Joint Distribution

Let $T$ be the lifetime of the item under consideration. In practice, informed judgement about $T$ is often available in the form of vendor information, engineering knowledge or experience in the field. We assume that reliability estimates $r^{(n)} = (r_1, r_2, \ldots, r_n)$ are provided for different mission times. $r_i$, $i = 1, 2, \ldots, n$, is considered an expert’s estimate of the reliability for mission time $t_i$. Let $r_{n+1}$ be a lower bound so that

$$1 \equiv r_0 \geq r_1 \geq r_2 \geq \ldots \geq r_n \geq r_{n+1} \geq 0, \quad 0 \leq t_1 \leq t_2 \leq \ldots \leq t_n < \infty$$

(1)

Assuming a Weibull model for the lifetime $T$ with reliability function $R(t/\lambda, \beta) = e^{-\lambda t^\beta}$, the likelihood model adopted for the expert’s data is the Dirichlet distribution [19], where

$$p(r_1, r_2, \ldots, r_n/\lambda, \beta) = \frac{\Gamma(b)}{\prod_{i=1}^{n+1} \Gamma(b_i)} \frac{\prod_{i=1}^{n+1} (r_i - r_{i-1})^{b_{i-1}}}{(1 - r_{n+1})^{b-1}}$$

(2)
and \(\{a_i\}_{i=1}^{n+1}\) are functions of \(\lambda\) and \(\beta\), and \(\sum_{i=1}^{n+1} a_i = 1\), \(b > 0\), \(a_i > 0\), \(i = 1, \ldots, n + 1\). The marginal distributions are

\[
p(r_i/\lambda, \beta) = \frac{\Gamma(b)(r_i - r_{n+1})^{b(1-a_{i*})-1}(1 - r_i)^{b a_{i*} - 1}}{(1 - r_{n+1})^{b-1}}
\]

(3)

with \(a_{i*} = \sum_{j=1}^{i} a_j\), means \(E(r_i/\lambda, \beta) = r_{n+1} + (1 - r_{n+1})(1 - a_{i*})\) and variances \(V(r_i/\lambda, \beta) = (1 - r_{n+1})^2 a_{i*}(1 - a_{i*})/(1 + b)\).

Given \(\lambda\) and \(\beta\), a perfect expert input would be \(e^{-\lambda t \beta i}\) for the reliability at time \(t_i\). Following Lindley [12] and introducing modulation parameters \(\mu(n) = (\mu_1, \mu_2, \ldots, \mu_n)\) and \(\sigma(n) = (\sigma_1, \sigma_2, \ldots, \sigma_n)\), we assume that \(E(r_i/\lambda, \beta) = \mu_i + \sigma_i e^{-\lambda t_i \beta} = e_i\) to obtain \(a_{i*} = (1 - e_i)/(1 - r_{n+1})\), \(a_i = (e_i - e_{i-1})/(1 - r_{n+1})\), \(i = 1, 2, \ldots, n\), \(e_0 = 1\).

To select \(b\), a variety of approaches can be used. The covariance structure of \(r^{(n)}\) is imposed by the choice of the Dirichlet model, and for large (small) values of \(b\), small (large) variances result. Aboura [1] determines \(b(\lambda, \beta)\) for the two cases where upper and lower bound functions of \((\lambda, \beta)\) are provided for the expert's estimates. In practice, it is often hard to obtain information about the spreads of the expert estimates. In that case the variances in the expert likelihood models are set to reasonable values and varied as part of the analysis. In the following, \(b\) is set to a constant.

The prior joint density distribution of \(\lambda\) and \(\beta\), given the expert's informed judgment \(r^{(n)}\), \(p(\lambda, \beta/r^{(n)})\), is obtained through the application of Bayes theorem:

\[
p(\lambda, \beta/r^{(n)}) \propto p(r^{(n)}/\lambda, \beta)p(\lambda, \beta)
\]

(4)

In practice it is often the case that no prior knowledge exists about \(\lambda\) and \(\beta\). In the following, we will assume a flat prior for \(\lambda\) and \(\beta\).

The prior density distribution \(p(\lambda, \beta/r^{(n)})\) is obtained numerically. Deriving the posterior distributions for inference requires numerical integration. Soland [17] introduced a mixed prior distribution for the parameters of the Weibull distribution that allows for closed form posterior distributions for failure and right-censored observations. \(\beta\) has a discrete distribution and \(\lambda\) a natural conjugate Gamma distribution. \(\lambda\) and \(\beta\) are assumed independent.

Mazzuchi and Soyer [13] used Soland’s distribution for the parameters of the Weibull lifetime distribution. A discretized Beta distribution is used for the parameter \(\beta\). Although such a use of Soland’s distribution does provide a starting prior joint density, one could dispute the feasibility of collecting any direct information from an expert about \(\beta\), the abstract model parameter \(\beta\) not having any physical meaning. One can also argue about the arbitrariness used
by Mazzuchi and Soyer [13] to select the range of the discretized Beta distribution for $\beta$, unless this range is made to cover most of the likely values of $\beta$. Here we remedy these shortfalls by constructing a prior density for $(\lambda, \beta)$ using estimates of observables. The range of $\beta$ and the dependence structure of $(\lambda, \beta)$ result naturally from the expert opinion elicitation procedure.

### 3.2 A Mathematically Tractable Prior

The distribution of Soland [17] is extended to include dependence and fitted through moments to the prior distribution $p(\lambda, \beta/r(n))$. The procedure to construct $g(\lambda, \beta/r(n))$, the fitted distribution, consists of the following four steps:

**Step 1.** Compute the prior distribution $p(\lambda, \beta/r(n))$.

**Step 2.** Determine the range of $\beta$. Let $[\beta_l, \beta_u]$ be the interval where most probability for $\beta$ falls, i.e.

$$\text{Prob}(\beta \in [\beta_l, \beta_u]/r(n)) = \int_{\beta_l}^{\beta_u} \int_0^\infty p(\lambda, \beta/r(n)) d\lambda d\beta \approx 1.$$  

**Step 3.** Discretize $\beta$. Let $k$ be a chosen number so that $\beta \in \{\beta_1, \beta_2, \ldots, \beta_k\}$, where $\beta_j = \beta_l + (j-1)(\beta_u - \beta_l)/(k-1)$, $j = 1, 2, \ldots, k$.

**Step 4.** Obtain $g(\lambda, \beta/r(n))$. For $j = 1, 2, \ldots, k$, compute $q_j$ the normalized value for $\text{Prob}(\beta = \beta_j/r(n))$ after discretization

$$q_j = \frac{\int_0^\infty p(\lambda, \beta_j/r(n)) d\lambda}{\sum_{j=1}^k \int_0^\infty p(\lambda, \beta_j/r(n)) d\lambda} \quad (5)$$

Compute the conditional mean

$$E_j = E(\lambda/\beta_j, r(n)) = \int_0^{\infty} \lambda p(\lambda, \beta_j, r(n)) d\lambda$$

and the conditional variance of $\lambda$, $V_j = V(\lambda/\beta_j, r(n))$. Let $c_j = E_j^2/V_j$ and $d_j = E_j/V_j$. The fitted conditional prior distribution for $\lambda$ given $\beta_j$ and $r(n)$ is Gamma$(c_j, d_j)$, i.e. $g(\lambda/\beta_j, r(n)) = d_j^{c_j} \lambda^{c_j-1} e^{-d_j \lambda} / \Gamma(c_j)$ where $g(\lambda, \beta_j/r(n)) = g(\lambda/\beta_j, r(n)) q_j$, for $j = 1, 2, \ldots, k$. The fitted prior distribution preserves the dependence structure
of $\lambda$ and $\beta$ contained in the original prior density. By construction it also preserves the conditional first two moments of $\lambda$, while doing the same for $\beta$ if $k$ is large enough.

Using the prior distribution $g(\lambda, \beta/r^{(n)})$, the first replacement time $T_1$ is determined based on costs and the expected number of failures in $[T_0, T_1]$. Subsequently, at each replacement time $T_{i-1}$, $i = 2, 3, \ldots$, the next replacement time $T_i$ is determined using costs and the expected number of failures in $[T_{i-1}, T_i)$, posterior to the observed failure and survival data in $[T_0, T_{i-1})$.

We begin by deriving the posterior distribution of $(\lambda, \beta)$ for the two data collection cases. We then derive the optimal replacement times.

### 3.3 The Posterior Distribution for Complete Data

Given the expert’s input $r^{(n)} = (r_1, r_2, \ldots, r_n)$ for $t^{(n)} = (t_1, t_2, \ldots, t_n)$ and an appropriate choice of $b$, the posterior distribution of $(\lambda, \beta)$ for observed failure times $f^{(m)} = (f_1, f_2, \ldots, f_m)$ and survival times (right censored observations) $s^{(l)} = (s_1, s_2, \ldots, s_l)$ is

$$
p(\lambda, \beta|f^{(m)}, s^{(l)}, r^{(n)}) \propto \frac{m}{\prod_{i=1}^{m} \lambda \beta f_1^{\beta-1} e^{-\lambda f_i}} \times \frac{l}{\prod_{u=1}^{l} e^{-\lambda s_u}} \times p(\lambda, \beta/r^{(n)})
$$

(6)

$p(\lambda, \beta/r^{(n)})$ is replaced by $g(\lambda, \beta/r^{(n)})$ (with parameters $c_j$, $d_j$ and $q_j$) to allow for a closed form of the posterior distribution. For $\beta_j = \beta_l + (j - 1)(\beta_u - \beta_l)/(k - 1)$, $j = 1, 2, \ldots, k$ and $0 \leq \lambda < \infty$, the joint posterior distribution follows from (6) with normalizing constant $\sum_{j=1}^{k} \int_{0}^{\infty} p(\lambda, \beta_j/f^{(m)}, s^{(l)}, r^{(n)})d\lambda$ to produce

$$
p(\lambda, \beta_j/f^{(m)}, s^{(l)}, r^{(n)}) =
\left[\prod_{i=1}^{m} f_i^{\beta_j-1}\right] q_j \lambda^{m+c_j-1} \exp\left[-\lambda(d_j + \sum_{i=1}^{m} f_i^{\beta_j} + \sum_{u=1}^{l} s_u^{\beta_j})\right] \Gamma(c_j) \times \frac{d_j^{c_j}}{\Gamma(d_j + \sum_{i=1}^{m} f_i^{\beta_j} + \sum_{u=1}^{l} s_u^{\beta_j})}
$$

(7)

The marginal posterior distribution of $\beta$, $\beta_j$, $p(\lambda, \beta_j/f^{(m)}, s^{(l)}, r^{(n)})d\lambda$, is

$$
p(\beta_j/f^{(m)}, s^{(l)}, r^{(n)}) =
$$
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\[
\beta_j^m \left\{ \prod_{i=1}^m f_j^{\beta_j - 1} \right\} d_j^{\Gamma(m + c_j)} \frac{d_j^j}{\Gamma(\sum_{i=1}^m f_j^{\beta_j} + \sum_{u=1}^l s_u^{\beta_j})} \frac{\prod_{i=1}^m f_j^{\beta_j - 1} \Gamma(\sum_{i=1}^m f_j^{\beta_j} + \sum_{u=1}^l s_u^{\beta_j})^{m+c_j}}{\sum_{j=1}^k \beta_j^m \left\{ \prod_{i=1}^m f_j^{\beta_j - 1} \right\} d_j^{\Gamma(m + c_j)} \frac{d_j^j}{\Gamma(\sum_{i=1}^m f_j^{\beta_j} + \sum_{u=1}^l s_u^{\beta_j})} \frac{\prod_{i=1}^m f_j^{\beta_j - 1} \Gamma(\sum_{i=1}^m f_j^{\beta_j} + \sum_{u=1}^l s_u^{\beta_j})^{m+c_j}}}
\]

(8)

The conditional posterior of \( \lambda \) is

\[
p(\lambda, \beta_j, f^{(m)}, s^{(l)}, r^{(n)}) / p(\beta_j, f^{(m)}, s^{(l)}, r^{(n)}) =
\]

\[
(\gamma_j^m + \sum_{i=1}^m f_j^{\beta_j} + \sum_{u=1}^l s_u^{\beta_j})^{m+c_j} \lambda^{m+c_j-1} \exp \left[ -\lambda \left( d_j^j \sum_{i=1}^m f_j^{\beta_j} + \sum_{u=1}^l s_u^{\beta_j} \right) \right]
\]

(9)

so that

\[
(\lambda, \beta_j, f^{(m)}, s^{(l)}, r^{(n)}) \sim \text{Gamma} \left( m + c_j, d_j^j + \sum_{i=1}^m f_j^{\beta_j} + \sum_{u=1}^l s_u^{\beta_j} \right).
\]

3.4 The Posterior Distribution for Interval Censored Data

For interval censored data \( m^{(i-1)} = (m_1, m_2, \ldots, m_{i-1}) \), where \( m_j \) is the observed number of failures in \([T_{j-1}, T_j)\), \( j = 1, 2, \ldots, i - 1 \), Aboura [2] determines the posterior distribution of \((\lambda, \beta)\) as

\[
p(\lambda, \beta/m^{(i-1)}) \propto \prod_{j=1}^{i-1} \left( \frac{M}{m_j} \right) (1 - e^{-\lambda \Delta T_j^\beta})^{m_j} (e^{-\lambda \Delta T_j^\beta})^{M-m_j} g(\lambda, \beta/r^{(n)})
\]

(10)

\( M \) being the total number of items and \( \Delta T_j = T_j - T_{j-1} \). For small \( i \) and \( M \), a closed form expression can be obtained for the posterior distribution through an extension of (10) in a Binomial sum. If \( i \) and \( M \) are realistically large, the number of summations in the closed form expression would become large and not practical to carry. A two-dimensional numerical computation provides \( p(\lambda, \beta/m^{(i-1)}) \).

4 The Replacement Strategy

The maintenance optimization procedure consists of determining at each planned replacement time \( T_{i-1} \), the next preventive replacement time \( T_i \),
$i = 1, 2, \ldots$. In between the prescribed times $T_1, T_2, \ldots$, replacement or repair are not made upon failure of the operating item. In the traditional maintenance optimization approach, the time intervals between planned replacements would be equal and determined at the start of the operations. In the approach presented here, the replacement times are determined a stage ahead. The adaptive nature of the policy reduces the economic loss of a fixed time replacement protocol, as the maintenance procedure reaches for an optimal replacement time. The expected cost per unit of time for the next replacement interval is minimized, the expectation taken with respect to past failure and survival information.

4.1 Cost Function

At time $T_{i-1}$, $i = 1, 2, \ldots$, the expected cost per unit of time in the interval $[T_{i-1}, T_i)$, is

$$c(\Delta T_i) = \frac{c_p + c_f E(N_i/D_{i-1})}{\Delta T_i} \quad (11)$$

where $c_p$ is the cost of replacing all $M$ items, $c_f$ is the cost of failure of one item and $E(N_i/D_{i-1})$ is the expected number of failures in $[T_{i-1}, T_i)$, the expectation taken with respect to the available information $D_{i-1}$. $D_0$ is the set of all relevant information known prior to and at time $T_0$. $D_0 = \{r^{(n)}, t^{(n)}, b\}$. In the case of complete data, $D_{i-1} = \{f^{(m)}, s^{(l)}\} \cup D_0$. $f^{(m)} = (f_1, f_2, \ldots, f_m)$ and $s^{(l)} = (s_1, s_2, \ldots, s_l)$ are the failure and survival times respectively, observed in $(T_0, T_{i-1})$. In the case of censored data, $D_{i-1} = \{m^{(i-1)}\} \cup D_0$, where $m^{(i-1)} = (m_1, m_2, \ldots, m_{i-1})$, $m_j$ being the observed number of failures in $[T_{j-1}, T_j)$, $j = 1, 2, \ldots, i-1$. The replacement cost $c_p$ is relatively easy to assess. $c_p$ consists of the cost of the $M$ new items and the cost of manhours needed for the replacement. $c_f$ is the cost due to the failure of one item. Although such a cost often exists in the form of inconvenience, reduced productivity or other, it may be hard to attach a dollar value to it. In Section 5, we consider several cost models.

4.2 Maintenance Constraint

If $c_p$ and $c_f$ are constants in (11), and since $N_i$, $i = 1, 2, \ldots$, the number of failed items in $[T_{i-1}, T_i)$, is bounded by $M$, $c(\Delta T_i)$ approaches 0 as $\Delta T_i$ becomes large. In practice, it makes sense to adopt the present maintenance scenario only if an acceptable proportion of items is to fail in $[T_{i-1}, T_i)$. To model this constraint, $c_f$ can be made to increase with the number of failed
items \( N_i \). This penalty function may be hard to assess. Instead, a constraint on the number of failures \( N_i \) can be introduced. For example, the probability of the number of failures \( N_i \) not exceeding a predetermined number \( K \) may be constrained to be at least \( 1 - \alpha, 0 < \alpha < 1 \). The problem parameters \( K \) and \( \alpha \) would be determined by the maintenance management. Another possible constraint on the number of failures \( N_i \) is to restrain the expected value \( E(N_i/D_i) \) to be less than some number \( K \). These constraints would force \( \Delta T_i \) to be bounded above, rendering the minimization of \( c(\Delta T_i) \) possible. In the next section, we consider four cases where (i) the cost of failure \( c_f \) is zero and the expected number of failures is constrained, (ii) the cost of failure \( c_f \) is a constant and the expected number of failures is constrained, (iii) the cost of failure \( c_f \) is a penalty function for the number of failures, and (iv) a penalty cost occurs if the number of failures exceeds a predetermined number.

5 Maintenance Optimization Models

5.1 The Constant Cost Model

The Constant Cost model considers the case in which there is no dollar value associated with the failure of an item \((c_f = 0)\). The range of possible time intervals before the next replacement is limited by a constraint on the expected number of failures in the considered interval. The maintenance optimization problem at time \( T_{i-1} \) is

\[
\begin{gathered}
\min_{\Delta T_i > 0} \frac{c_p}{\Delta T_i} \text{ subject to } E\left(\frac{N_i}{D_{i-1}}\right) \leq K
\end{gathered}
\]

In both data collection cases \( E\left(\frac{N_i}{D_{i-1}}\right) \leq K \) translates into \( R(\Delta T_i/D_{i-1}) \geq 1 - K/M \), where \( R(t/D_{i-1}) \) is the reliability for mission time \( t \) as assessed at time \( T_{i-1} \). Since

\[
E\left(\frac{N_i}{D_{i-1}}\right) = \sum_{j=1}^{k} \int_{0}^{\infty} E(N_i/\lambda, \beta_j) g(\lambda, \beta_j/D_{i-1}) d\lambda, \quad j = 1, 2, \ldots, k,
\]

\[
E(N_i/\lambda, \beta_j) = M(1 - e^{-\lambda \Delta T_i^{\beta_j}}) = M(1 - R(\Delta T_i/\lambda, \beta_j))
\]

and

\[
E(N_i/D_{i-1}) = M(1 - R(\Delta T_i/D_{i-1}))
\]

Therefore the optimal time interval is the largest feasible \( \Delta T_i \) value. The optimal time intervals \( \Delta T_i, i = 1, 2, \ldots \), depend only on the ratio \( K/M \).
Figure 1 The initial reliability estimates \( (t^{(5)}, r^{(5)}) \) plotted against \( e^{-2t^3} \).

and the reliability function \( R(\cdot / D_{i-1}) \). If \( \lambda \) and \( \beta \) were to be known, then
\[
R(\Delta T_i / D_{i-1}) = R(\Delta T_i / D_{\infty}) = e^{-\lambda \Delta T_i^\beta}, \quad i = 1, 2, \ldots, \quad \text{and} \quad \Delta T_{\infty} = (-\ln(1 - K/M)/\lambda)^{1/\beta}
\]
would be the optimal time interval between replacements, at all stages. \( \Delta T_{\infty} \) is the value to which \( \Delta T_i \) ultimately converges to in both data collection, given the assumption of a Weibull lifetime model.

5.1.1 Simulation of the Maintenance Optimization Procedure

In a simulated example where \( \lambda = 2, \beta = 3 \) and \( K/M = 0.2 \), we assume that reliability estimates \( r^{(5)} = (0.99, 0.95, 0.60, 0.05, 0.01) \) for mission times \( t^{(5)} = (0.2, 0.4, 0.75, 1, 1.2) \) are given by an expert or taken from some other knowledgeable source. Figure 1 shows the expert input \( (t^{(5)}, r^{(5)}) \) plotted against the reliability function \( e^{-2t^3} \). The resulting prior distribution \( g(\lambda, \beta / r^{(n)}) \) [2] is shown in Figure 2.

The optimal value for the first replacement time \( T_1 \) is obtained as the time \( t \) at which the prior reliability \( R(t / D_0) \) is equal to \( 1 - K/M = 0.8 \). Therefore in this example the first replacement of all items is to occur at time \( T_1 = \Delta T_1 = 0.265 \). The prior reliability function \( R(t / D_0) \) is shown in Figure 3 with the resulting optimal first time interval \( \Delta T_1 = 0.265 \). The
The prior distribution $g(\lambda, \beta/r^{(m)})$.

The dashed line function in the graph of Figure 3 is $e^{-2t^3}$ with the corresponding limiting optimal time interval $\Delta T = (-\ln(1 - 0.2)/2)^{1/3} = 0.481$.

At the successive times $T_{i-1}$, $i = 2, 3, \ldots$, the optimal time intervals $\Delta T_i$ are obtained as the solution of $R(\Delta T_i/D_{i-1}) = 1 - K/M$. In the case of complete data $R(\Delta T_i/D_{i-1})$ obtains in a closed algebraic form while it must be numerically evaluated at each stage, in the case of interval censored data [2].

**Complete Data**

$$R(\Delta T_i/D_0) = \sum_{j=1}^{k} \int_{0}^{\infty} e^{-\lambda \Delta T_i^p} g(\lambda, \beta_j/D_{i-1}) d\lambda$$

$$= \sum_{j=1}^{k} \left( \int_{0}^{\infty} e^{-\lambda \Delta T_i^p} \frac{d_j^\beta \lambda^{\beta-1} e^{-d_j \lambda}}{\Gamma(c_j)} d\lambda \right) q_j$$

$$= \sum_{j=1}^{k} \frac{d_j^\beta q_j}{(d_j + \Delta T_i^p)^{c_j}}$$
Figure 3 The first replacement time $\Delta T_1 = 0.265$ and the limiting optimal interval $\Delta T_\infty = 0.481$.

\[
\times \left( \int_0^\infty \frac{(d_j + \Delta T_1^{(p)})\lambda^{c_j - 1} e^{-\lambda(d_j + \Delta T_1^{(p)})}}{\Gamma(c_j)} d\lambda \right)
\]

\[
= \sum_{j=1}^k \left( \frac{d_j}{d_j + \Delta T_1^{(p)}} \right)^{c_j} q_j
\]

For $i = 2, 3, \ldots$, $R(\Delta T_i / D_{i-1}) = R(\Delta T_i / D_0, f^{(m)}, s^{(l)})$, and

\[
R(\Delta T_i / D_0, f^{(m)}, s^{(l)})
\]

\[
= \sum_{j=1}^k \left( \int_0^\infty e^{-\lambda \Delta T_i^{(p)}} p(\lambda / \beta_j, f^{(m)}, s^{(l)}) d\lambda \right)
\]

\[
= \sum_{j=1}^k \left( \int_0^\infty e^{-\lambda \Delta T_i^{(p)}} \frac{d_j + \sum_{i=1}^m f_i^{\beta_j} + \sum_{u=1}^l s_u^{\beta_j}}{\Gamma(m + c_j)} d\lambda \right)
\]

\[
\times \lambda^{m+c_j-1} \exp \left[ -\lambda \left( d_j + \sum_{i=1}^m f_i^{\beta_j} + \sum_{u=1}^l s_u^{\beta_j} \right) \right] d\lambda
\]
Adaptive Maintenance Optimization Using Initial Reliability Estimates

The optimal time intervals \( \Delta T_i, i = 1, 2, \ldots, 20 \) for \( M = 10, K = 2 \) and \( c_p = 30 \), are plotted in Figure 4 for a 20 stages simulation of the maintenance routine. In this example \( M = 10, K = 2 \) and \( c_p = 30 \). The exact failure times are recorded between the replacement times. The horizontal dashed line in Figure 4 marks the limiting optimal time interval \( \Delta T_\infty = 481 \). As data is gathered between the replacement times, the optimal time intervals improve to finally stabilize around the limiting value. Figure 5 shows the prior distribution and the posterior distribution of \((\lambda, \beta)\) at time \( T_{19} \).

Figures 6 and 7 show the optimal time intervals \( \Delta T_i, i = 1, 2, \ldots, 10 \), for \( M = 40, K = 8 \), and \( M = 100, K = 20 \), respectively.

The adaptive nature of the maintenance procedure is observed in the two examples. Similar behavior of the maintenance optimization procedure was observed in the case of interval censored data collection.

To study the convergence of the adaptive procedure, 100 simulation runs were made for both data collection cases. Each replication consisted of

\[
p(\beta_j/f^{(m)}, s^{(l)}) = \sum_{j=1}^{k} \frac{(d_j + \sum_{i=1}^{m} f_i^{(j)} + \sum_{u=1}^{l} s_u^{(j)} \beta_j^{m+c_j})^{m+c_j}}{(d_j + \sum_{i=1}^{m} f_i^{(j)} + \sum_{u=1}^{l} s_u^{(j)} + \Delta T_i^{(j)})^{m+c_j}} p(\beta_j/f^{(m)}, s^{(l)})
\]
Figure 5 Prior and posterior distribution of \((\lambda, \beta)\) at time \(T_{19}\).

Figure 6 The optimal time intervals \(\Delta T_i\), \(i = 1, 2, \ldots, 20\) for \(M = 40, K = 8\).
of the simulation of 10 replacement stages for $M = 100$, $K = 20$. Figure 8 shows the maximum, mean and minimum optimal time interval $\Delta T_i$, $i = 1, 2, \ldots, 10$ for the case of complete data observation, the statistics computed over the 100 replications. Figure 9 shows the equivalent of Figure 8 for the interval censored data collection scenario. In both cases, the maintenance optimization procedure converges rapidly in the mean value, with the spread around the mean values getting smaller as more information is gathered through the replacement stages. The case of interval censored data shows more spread around the mean values, as expected.

Note that although the expert reliability estimates indicate an optimistic opinion for the early life of the item, the first optimal replacement time $\Delta T_1 \approx 0.265$ is a cautious decision, being smaller than 0.481. This is due to the fact that the prior variance parameter $b$ has been set to 0.001. This choice of value for $b$ gives a prior mean of 1.506 for $\lambda$ and a prior mean of 1.576 for $\beta$ with relatively large prior variances. If $b$ was made very large, a prior distribution degenerate at (2.926, 5.831) would have resulted for $(\lambda, \beta)$, with a first replacement time equal to 0.643. The smaller $b$, the more variance is allowed in the prior distribution providing a better adaptation to observed data. Table 1 shows the prior means and variances of $\lambda$ and $\beta$ for different values of $b$. 
Figure 8 The maximum, mean and minimum optimal time interval $\Delta T_i, i = 1, 2, \ldots, 10$ for the case of complete data observation.

Figure 9 The maximum, mean and minimum optimal time interval $\Delta T_i, i = 1, 2, \ldots, 10$ for the case of the interval censored data.
Table 1 Prior means and variances of $\lambda$ and $\beta$ for different values of the parameter $b$.

<table>
<thead>
<tr>
<th>$b$</th>
<th>$10^{-10}$</th>
<th>$10^{-9}$</th>
<th>1</th>
<th>10</th>
<th>100</th>
<th>10$^4$</th>
<th>$2 \times 10^4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda$ Mean</td>
<td>1.506</td>
<td>1.506</td>
<td>1.615</td>
<td>2.002</td>
<td>2.478</td>
<td>2.854</td>
<td>2.923</td>
</tr>
<tr>
<td>$\lambda$ Variance</td>
<td>0.377</td>
<td>0.377</td>
<td>0.347</td>
<td>0.198</td>
<td>0.070</td>
<td>0.015</td>
<td>0.001</td>
</tr>
<tr>
<td>$\beta$ Mean</td>
<td>1.576</td>
<td>1.576</td>
<td>1.854</td>
<td>3.067</td>
<td>4.677</td>
<td>5.660</td>
<td>5.823</td>
</tr>
<tr>
<td>$\beta$ Variance</td>
<td>0.372</td>
<td>0.372</td>
<td>0.396</td>
<td>0.362</td>
<td>0.236</td>
<td>0.054</td>
<td>0.006</td>
</tr>
</tbody>
</table>

5.2 The Constant Failure Cost Model

The Constant Failure Cost model considers the case in which a constant cost $c_f$ is associated with the failure of each item. The range of possible time intervals before the next replacement remains limited by a constraint on the expected number of failures in the considered interval. The maintenance optimization problem at time $T_{i-1}$ is

$$\min_{\Delta T_i > 0} \frac{c_p + c_f E(N_i/D_{i-1})}{\Delta T_i}$$

subject to $E(N_i/D_{i-1}) \leq K \quad (13)$

Figure 10 shows the first optimal interval $\Delta T_1 = 0.265$ and the optimal time interval given knowledge of $\lambda = 2, \beta = 3, \Delta T_\infty = 0.361$ for the example of Section 5.1 when $c_p = 30$ and $c_f = 18$. The full line in Figure 10 is the cost function of (13) for $i = 1$. The dashed line is the limiting cost function $(c_p + c_f M(1 - R(t/D_\infty))/t = (c_p + c_f M(1 - e^{-2t}))/t$. Given the positive cost of failure in this model, the limiting optimal time interval $\Delta T_\infty = 0.361$ is less than its value 0.481 obtained in the case of the Constant Cost model. As failure information is gathered, the cost function approaches its limit shown by the dashed line in Figure 10, making the optimal time interval $\Delta T_i$ converge to its limiting value $\Delta T_\infty = 0.361$.

5.3 The Penalty Cost Model I

This penalty cost model assumes an increasing cost function $c_f(j)$ for the failure of $j$ items. Various functions may be chosen to model $c_f(j)$ depending on the application. The maintenance optimization problem at time at time $T_{i-1}$ is

$$\min_{\Delta T_i > 0} \frac{c_p + \sum_{j=0}^{M} c_f(j) \operatorname{Prob}(N_i = j/D_{i-1})}{\Delta T_i} \quad (14)$$

where

$$\operatorname{Prob}(N_i = j/D_{i-1})$$
Although (14) is an unconstrained minimization problem, it can be seen that for an appropriate choice of the penalty function $c_f$, an optimal solution $\Delta T_i$ obtains for each stage $i = 1, 2, \ldots$, etc. The determination of the penalty function $c_f(\cdot)$ may not be a trivial task and therefore makes this model hard to apply.

### 5.4 The Penalty Cost Model II

A more applicable penalty cost model is a special case of that of Section 5.3, in which a penalty is applied when the number of failures exceeds a predetermined number. An example is when a city council imposes such a penalty on the electricity company for not keeping a certain percentage of the city lights operating in a particular area of the city. In this case, the maintenance optimization problem at time $T_{i-1}$ is

$$
\min_{\Delta T_i > 0} \frac{c_p + c_f \Prob(N_i > K / D_{i-1})}{\Delta T_i}
$$

(16)
For (16) to have a solution, a constraint needs to be imposed, say $\Delta T_i \leq \delta$ for some $\delta$ to be defined by the maintenance management.

6 Concluding Remarks

An adaptive procedure for the optimal replacement of identical items operating under similar conditions was outlined. Four maintenance optimization models were developed. Two data collection scenarios were considered. The procedure was demonstrated and its convergence shown in both data collection cases in simulated examples. The procedure is easy to implement and can result in substantial savings. The adaptive nature of the procedure is a modern feature that permits an updating of the lengths of times between replacements as failure information is gathered. The methodology in this paper was developed following technical discussions with an electricity company.

References

Biographies

Khalid Aboura teaches quantitative methods at the College of Business Administration, University of Dammam, Kingdom of Saudi Arabia. Dr. Aboura spent several years involved in academic research and consulting at the George Washington University, Washington DC, USA, where he completed the Master of Science and the Doctor of Science degrees in Operations Research. Dr. Aboura has extensive experience in Stochastic Modelling in Operations Research and Engineering, Simulation, Maintenance Optimization and Mathematical Optimization. He served as Chairman of the Statistical Computing Section of the Washington Statistical Society. Khalid Aboura worked as a Research Scientist at the Division of Mathematics and Statistics of the Commonwealth Scientific and Industrial Research Organization (CSIRO) of Australia. During his tenure at CSIRO, Khalid Aboura was involved in research and consulting with the Australian industry on a number of projects. Khalid Aboura also conducted research...
Johnson I. Agbinya is currently Associate Professor in the department of electronic engineering at La Trobe University, Melbourne Australia. He is also Honorary Professor at the University of Witwatersrand (WITS), South Africa; Extraordinary Professor at the University of the Western Cape (UWC), Cape Town and the Tshwane University of Technology (TUT), Pretoria, South Africa. Prior to joining La Trobe University in November 2011, he was Senior Research Scientist at CSIRO Telecommunications and Industrial Physics (now CSIRO ICT) from 1993–2000, Principal Research Engineering at Vodafone Australia (2000–2003) and Senior Lecturer at UTS Australia (2003–2011). His R&D activities cover remote sensing, Internet of things (machine to machine communications), bio-monitoring systems, wireless power transfer, mobile communications and biometrics systems. He has authored/co-author nine technical books in telecommunications, some of which are used as textbooks. He is founder of the International conference on broadband communications and biomedical applications (IB2COM), Pan African conference on science, computing and telecommunications (PACT) and the African Journal of Information and Communication Technology (AJICT). He has published more than 250 peer-reviewed research publications in international Journals and conference papers. He received his BSc degree electronic/electrical engineering from Obafemi Awolowo University (OAU), Ile Ife, Nigeria; MSc in electronic engineering from the University of Strathclyde, Glasgow Scotland and PhD from La Trobe University, Melbourne, Australia in 1973, 1982 and 1994 respectively. He received Best Paper award from IEEE 5th International Conference on Networking (ICN’ 2006) Mauritius, CSIRO ADCOM group research award in 1997 and Research Trailblazer Certificate at UTS in 2009. He is the Editor in Chief of the African Journal of ICT (AJICT), General Chair of several international conferences and member of several current international technical conference committees. He has served as expert on several international grants reviews/committees and was a rated researcher by the South African National Research Fund (NRF).