Chapter 12
Modelling a Nonlinear Single-Degree-of-freedom System from Experimental Data using Lagrangian Neural Networks

Alan Xavier and Ludovic Renson

Abstract Accurately modelling the dynamic behaviour of nonlinear structures is challenging due to the wide range of potential nonlinearities and dynamic phenomena that they can exhibit. Physics-guided machine learning (PGML) has emerged as an attractive way to combine prior knowledge with data to solve a wide range of complex, nonlinear problems in science and engineering. Lagrangian Neural Networks (LNNs) are a particular PGML approach that models nonlinear systems' Lagrangian functions using artificial neural networks (NNs). The Euler-Lagrange equation is then reconstructed through automatic differentiation (AD) to derive the equations of motion, enforcing physical consistency during training.

In this work, we use LNNs to model the nonlinear vibrations of mechanical structures. We now explore the applicability of our methodology to a nonlinear SDOF system from experimental data. We analyse the physical consistency of the trained model and interpret the identified stiffness and damping nonlinearities from the partial derivatives of the potential energy and dissipation functions.

Keywords Physics-guided Machine Learning · Lagrangian Neural Networks · Control-Based Continuation

Introduction

Physics-guided machine learning (PGML) combines physics principles and domain knowledge with data-driven approaches. The goal of PGML is to create models that outperform both physics-based and data-driven ML models when used in isolation, and address common limitations. Cranmer et al. [1] proposed Lagrangian Neural Networks (LNNs), where the Lagrangian, $\mathscr{L} = \mathscr{T} - \mathscr{V}$, is parameterized by a neural network to model the action of a mechanical system, i.e. $\mathscr{L} \approx \mathscr{L}_\theta$, where θ are trainable weights. By applying Hamilton's principle, the Euler-Lagrange equation

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathscr{L}}{\partial q_j} = 0 \tag{1}$$

provide the system's equations of motion (EoM). A convenient vector form of this equation (1) [1] is.

$$\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_j} = \frac{\partial \mathscr{L}}{\partial q_j} \quad \rightarrow \quad \frac{d}{dt}\nabla_{\dot{q}_j}\mathscr{L} = \nabla_q \mathscr{L} \quad \rightarrow \quad \left(\nabla_{\dot{q}}\nabla_{\dot{q}}^T \mathscr{L}\right)\ddot{q} + \left(\nabla_q \nabla_{\dot{q}}^T \mathscr{L}\right)\dot{q} = \nabla_q \mathscr{L} \tag{2}$$

The resulting EoM can then be used to solve for $\ddot{q}$ (3):

$$\ddot{q} \quad = \quad \left(\nabla_{\dot{q}}\nabla_{\dot{q}}^T \mathscr{L}_\theta\right)^{-1}\left[\nabla_q \mathscr{L}_\theta - \left(\nabla_q \nabla_{\dot{q}}^T \mathscr{L}_\theta\right)\dot{q}\right] \tag{3}$$

where $\mathscr{L}_\theta$ is the Lagrangian parameterized by a neural network. The neural network parameters can be obtained through the minimisation of a loss function comparing measured data, $\ddot{q}_{true}$, and model predictions, $\ddot{q}_{\mathscr{L}NN}$.

To model vibrating structures, the idea of LNNs was extended to deal with systems with external excitation and non-conservative forces [2]. Incorporating an external excitation $F(t)$ and a dissipation function $\mathscr{D}$ in (1), using the method described in [3], yields the modified Euler-Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial \mathscr{L}}{\partial \dot{q}_j}\right) - \frac{\partial \mathscr{L}}{\partial q_j} + \frac{\partial \mathscr{D}}{\partial \dot{q}_j} - \mathscr{F}(t) = 0 \tag{4}$$

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The loss function used to minimize any discrepancy between the model and $\ddot{q}_{true}$ is now given by

$$\hat{\ddot{q}} = \left(\nabla_{\dot{q}} \nabla_{\dot{q}}^T \mathscr{L}_\theta\right)^{-1} \left[\nabla_q \mathscr{L}_\theta - \left(\nabla_q \nabla_{\dot{q}}^T \mathscr{L}_\theta\right) \dot{q} - \nabla_{\dot{q}} \mathscr{D}_\theta + \mathscr{F}(t)\right], \tag{5}$$

where the damping and conservative dynamics of the system are modelled using separate neural networks, $\mathscr{D}_\theta$ and $\mathscr{L}_\theta$, respectively. Additional constraints are then introduced for the system at rest, namely that $\mathscr{L}(0,0) = 0$ and $\mathscr{D}(0) = 0$. The loss function can then be written as:

$$loss = \lambda_0 \cdot ||\hat{\ddot{q}} - \ddot{q}|| + \sum_{i=1}^{n} \lambda_i \cdot \mathscr{G}_i, \tag{6}$$

where $\mathscr{G}_i$ is the i-th constraint in the loss function with the associated i-th weight, λ_i, balancing the loss terms. This allows us to enforce any prior knowledge about the system to improve our network's performance. The method was successfully demonstrated numerically on a Duffing oscillator in [2].

Here, we test this LNN methodology on the experimental data collected on an SDOF oscillator with an adjustable softening-hardening restoring force. The experimental setup is shown in figure 1. The training data was then obtained using Control-based Continuation (CBC). Details on the experimental setup and the CBC algorithms used can be found in [4]. The experimental data set is a collection of so-called S-curves measured at fixed excitation frequencies, ω, between 23.7 Hz and 25.4 Hz in steps of 0.1 Hz. In total, 780 periodic responses were measured, each consisting of a time series with 2000 samples. The displacements, x, y, velocities, $\dot{x}, \dot{y}$ and accelerations, $\ddot{x}, \ddot{y}$, for the mass and base respectively, were directly available from the experimental data. We divide the 780 continuation samples into 700 training and 80 test samples, as shown in figure 2a.

Compared to our previous application of LNNs, the experimental system considered here has a base excitation. Assuming that x is the displacement of the mass and y is the displacement of the base, we can write the EoM as

$$m\ddot{x} + c[\dot{x} - \dot{y}] + k[x - y] + f_{\text{NL}}(x - y, \dot{x} - \dot{y}) = 0, \tag{7}$$

where m, k, c are the mass, stiffness and damping coefficients respectively. We further simplify (7) by letting $q = x - y$ and dividing throughout by m as shown in equation (8):

$$m[\ddot{x} - \ddot{y}] + c[\dot{x} - \dot{y}] + k[x - y] + f_{\text{NL}}(x - y, \dot{x} - \dot{y}) = -m\ddot{y} \quad \Rightarrow \quad \ddot{q} + \frac{c}{m}\dot{q} + \frac{k}{m}q + \frac{1}{m} f_{\text{NL}}(q, \dot{q}) = -\ddot{y} \tag{8}$$

The prediction of the accelerations from the Lagrangian, $\mathscr{L}_\theta(q, \dot{q})$, and damping, $\mathscr{D}_\theta(\dot{q})$, functions is thus given by

$$\hat{\ddot{q}} = \left(\nabla_{\dot{q}} \nabla_{\dot{q}}^T \mathscr{L}_\theta\right)^{-1} \left[\nabla_q \mathscr{L}_\theta - \left(\nabla_q \nabla_{\dot{q}}^T \mathscr{L}_\theta\right) \dot{q} - \nabla_{\dot{q}} \mathscr{D}_\theta - \ddot{y}\right]. \tag{9}$$

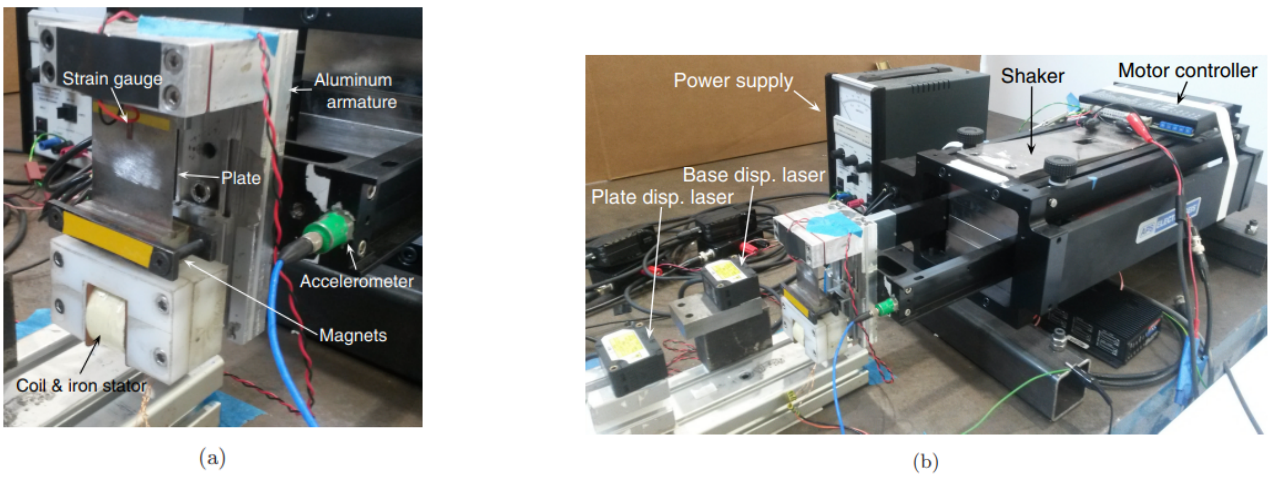
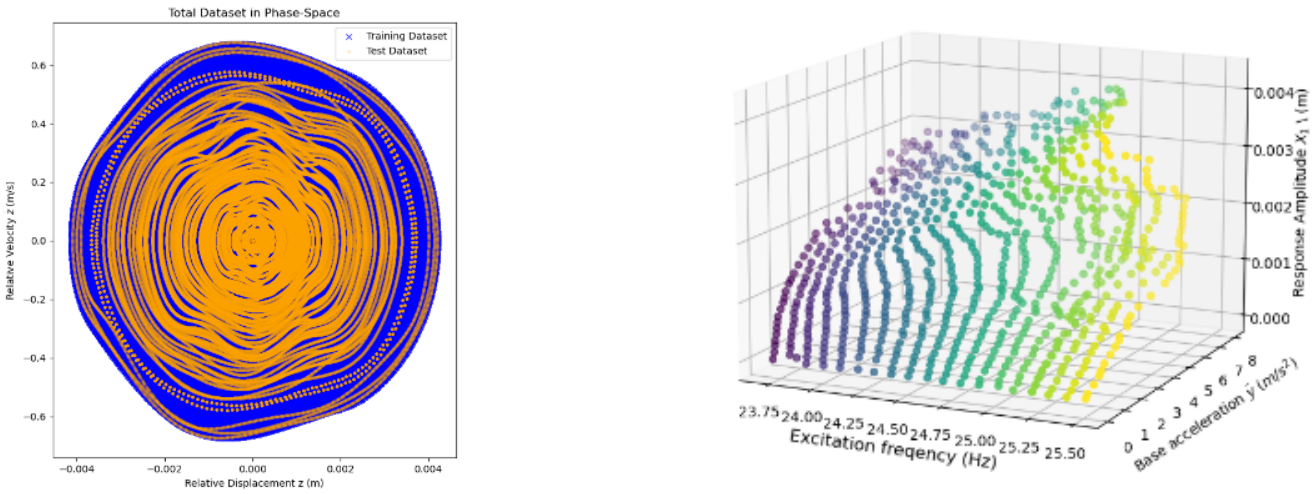


Fig. 1 (a) Picture of the nonlinear oscillator and (b) experimental setup for the SDOF nonlinear system with base excitation [4].



(a) (b)

Fig. 2 (a) Experimental data in phase-space. Blue (x)'s represent the training data, while the orange (•) represents our test data. (b) Forced response of the system, where the (•) are the amplitude of steady-state periodic responses measured during a series of continuations in amplitude (S-curves).

Our LNN is made up of two distinct NNs, one for the Lagrangian, $\mathscr{L}_\theta(q, \dot{q})$, and the other for the damping function, $\mathscr{D}_\theta(\dot{q})$. Each network has its own set of adjustable parameters, θ_i, and they were trained simultaneously to compute equations (5).

Both $\mathscr{L}_\theta$ and $\mathscr{D}_\theta$ have 16 hidden units across 2 hidden layers and use the SiLu activation function. The output layer does not use an activation function. Similarly, no physical bias was used during training. A normalizing layer was included in this network to enhance training stability, and was not learnt during training, but depends only on the data available. The model was trained for 100 epochs, with the Adam optimizer and a learning rate of 10^{-3}.

In figure 3, we present a comparison of the predicted relative acceleration, $\hat{\ddot{q}}$, with the true acceleration, $\ddot{q}$, across different excitation conditions ($\ddot{y}$) within the test dataset. This comparison reveals a qualitatively good agreement between the predicted and true values. However, the LNN fails to capture the higher harmonics in the signal.

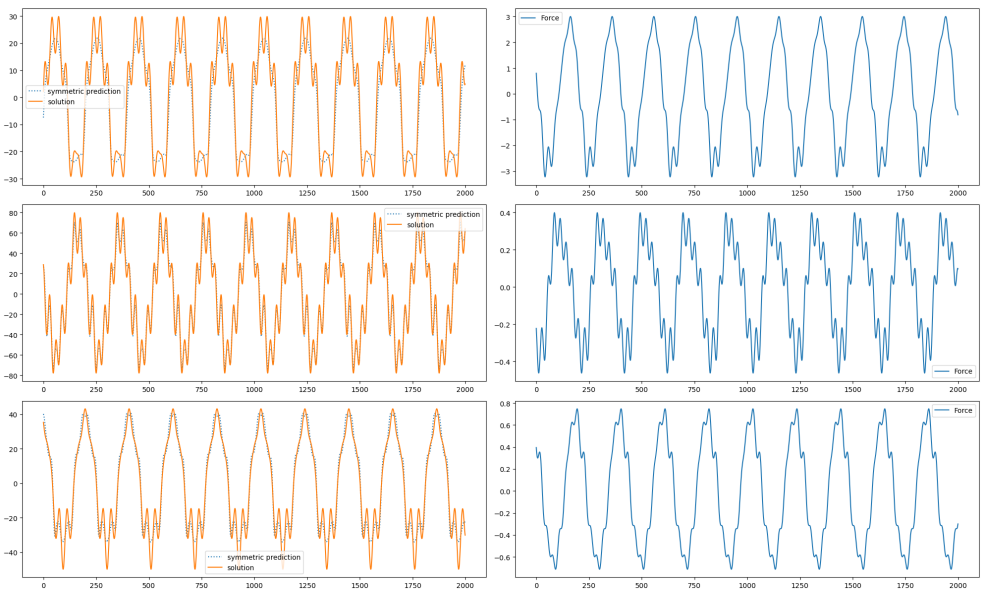


Fig. 3 The predicted relative acceleration, $\hat{\ddot{q}}$, vs the ground truth acceleration, $\ddot{q}$, by the LNN (left) and the applied excitation $\ddot{y}$.

Conclusions and Future Work

We showed that our modified LNN is capable of constructing equations of motion for a base-excited SDOF system from experimental data obtained using CBC. More specifically, the obtained model provides a good qualitative agreement without any knowledge of the system's physical properties, such as mass, stiffness and damping, but fails to reproduce the higher harmonics visible in the true dynamics.

In the future, we will use the LNNs trained on experimental data to trace the frequency response and bifurcation curves for the SDOF model. We will then compare these curves to the ones obtained using the data obtained from experiments. To improve the physical consistency of the LNN, separating NNs for predicting the kinetic and potential energies will also be assessed to address the dependence on different input features.

References

1. Cranmer, M.D., Greydanus, S., Hoyer, S., Battaglia, P.W., Spergel, D.N., and Ho, S. "Lagrangian neural networks". *CoRR*, abs/2003.04630 (2020)
2. M. J. Szydlowski, C.S. and Renson, L. "Modeling nonlinear structures using physics-guided, machine-learnt models". In *Proceedings of the 41st IMAC, A Conference and Exposition on Structural Dynamics 2023* (2023)
3. Meirovitch, L. *Methods of Analytical Dynamics*. Advanced engineering series. Dover Publications (2010)
4. Renson, L., Barton, D.A.W., and Neild, S.A. "Experimental tracking of limit-point bifurcations and backbone curves using control-based continuation". *International Journal of Bifurcation and Chaos*, 27(01):1730002 (2017)