Chapter 13
Computing Periodic Responses of Highly Flexible Beams Modelled under the SE(3) Lie Group Framework with Multiple Shooting

Amir K. Bagheri, Valentin Sonneville, and Ludovic Renson

Abstract Flexible mechanical structures with large displacements and rotations exhibit geometric nonlinearities and require accurate numerical modelling. Geometrically exact beam theory addresses these challenges by maintaining strain measures consistent with equilibrium equations in deformed configurations with finite rotations. However, standard finite element models using the $SO(3)$ Lie group face challenges with rotation parametrisation, non-commutative behaviour, and shear locking, leading to inaccuracies.

Beam formulations based on the Special Euclidean Lie Group $SE(3)$ resolve these issues by coupling rotations and positions with a local frame approach. This ensures strain invariance under rigid body motion and eliminates shear locking through a nonlinear finite element interpolation using the exponential map.

This work computes nonlinear normal modes of 3D beams modelled under the $SE(3)$ framework, applying multiple shooting and pseudo-arclength continuation methods. The multiple shooting method, overcomes limitations of the single shooting method, such as limited domain of attraction and high sensitivity to initial conditions. By discretising the periodic orbit into partitions and integrating over shorter intervals, it minimises error propagation and reduces branch jumps, particularly in systems with Unstable Periodic Orbits (UPOs). Additionally, parallel implementation of this method offers significant computational time savings, making it ideal for large finite element models.

Keywords $SE(3)$ Lie Group · Multiple Shooting · Geometric Nonlinearity · Nonlinear Normal Modes · Continuation

Introduction

The single shooting method is a common approach for solving boundary value problems, valued for its simplicity and ease of implementation alongside existing simulation codes. It also allows for the computation of stability via the monodromy matrix. However, it has notable drawbacks, including a limited domain of attraction for unstable orbits and a tendency for error growth over long integration periods. This can lead to difficulties in locating UPOs and may result in branch jumps when solution branches exist in close proximity, as highlighted by Lust [1] and Lakshmi [2].

To address these challenges, Keller [3] introduced the multiple shooting method, which discretises periodic orbits into smaller partitions, enabling separate time simulations for each partition. This strategy increases the domain of attraction, minimises error propagation, and reduces the likelihood of branch jumps. The method also enhances accuracy in detecting UPOs and can handle rapidly diverging systems more effectively. Parallel implementation further offers significant computational time savings, making it ideal for large multi-degree-of-freedom finite element models.

The multiple shooting method has garnered attention in various fields. Early work by Cebeci and Keller [4] demonstrated its potential to reduce sensitivity to initial conditions, improving solution stability. Bock and Plitt [5] applied it to optimal control problems, while Al-Mdallal et al. [6] showed its efficiency in solving fractional boundary value problems. Gonzales

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and James [7] employed it for computing invariant manifolds in discrete-time systems, demonstrating computational savings, and Artola et al. [8] showcased its utility in nonlinear optimal control of flexible structures. Finally, Schäfer et al. [9] used a variation of the method for nonlinear model predictive control, achieving notable reductions in computational time.

This paper builds on these advancements, applying the multiple shooting method to 3D beam formulations in the Special Euclidean Lie Group $SE(3)$ framework to enhance computational efficiency and accuracy in modelling flexible mechanical structures. The method is used to compute nonlinear normal modes, providing insights into the dynamic behaviour of highly flexible beams under geometric nonlinearities.

Multiple Shooting Formulation

We denote the result of the time integration of the system's equations of motion, over an interval of length T and from an initial condition vector $\boldsymbol{x}_0$ as

$$\phi(\boldsymbol{x}_0, T),\ \phi : \mathbb{R}^M \times \mathbb{R} \to \mathbb{R}^M, \tag{1}$$

We now consider s as a partition of the unit interval such that

$$0 = s_0 < s_1 < \cdots < s_{m-1} < s_m = 1, \tag{2}$$

with

$$\Delta s_i = s_i - s_{i-1}, \tag{3}$$

Note that if the unit interval is split into equal partitions, we have $\Delta s = \frac{1}{m}$ for all values of i. Multiple shooting then solves the following nonlinear system of m equations for finding a periodic orbit

$$\begin{cases} \boldsymbol{\psi}_0 = \phi(\boldsymbol{x}_0, \Delta s_1 T) - \boldsymbol{x}_1 = 0, \\ \boldsymbol{\psi}_1 = \phi(\boldsymbol{x}_1, \Delta s_2 T) - \boldsymbol{x}_2 = 0, \\ \vdots \\ \boldsymbol{\psi}_{m-2} = \phi(\boldsymbol{x}_{m-2}, \Delta s_{m-1} T) - \boldsymbol{x}_{m-1} = 0, \\ \boldsymbol{\psi}_{m-1} = \phi(\boldsymbol{x}_{m-1}, \Delta s_m T) - \boldsymbol{x}_0 = 0 \end{cases} \tag{4}$$

This is shown schematically in Figure 1, which depicts a periodic orbit split into m partitions over the period. For the orbit to be periodic, the time integration from the starting point of each partition and over the length of the partition should bring the system to the starting point of the next partition, within the tolerance specified. For the final partition, the time integration should bring the system back to the first partition in order to close the orbit. To find a periodic solution, the sensitivities of

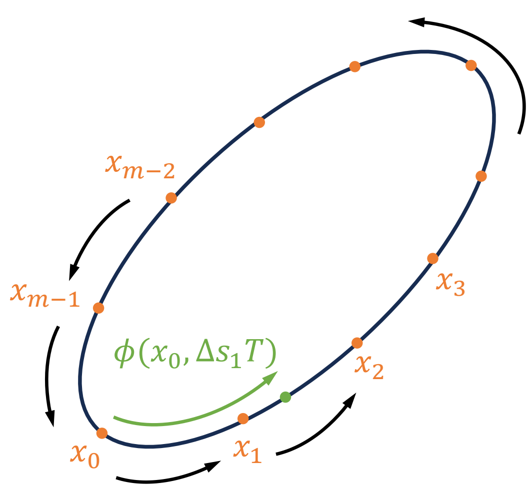


Fig. 1 Schematic illustration of the multiple shooting method.

the periodicity equations with respect to the initial conditions are computed and stored in the following Jacobian matrix:

$$\begin{bmatrix} \mathbf{M}_0 & -\mathbf{I} & & & & \mathbf{b}_0 \\ & \mathbf{M}_1 & -\mathbf{I} & & & \mathbf{b}_1 \\ & & \mathbf{M}_2 & -\mathbf{I} & & \mathbf{b}_2 \\ & & & \ddots & & \vdots \\ -\mathbf{I} & & & & \mathbf{M}_{m-1} & \mathbf{b}_{m-1} \\ \mathbf{B}_0 & \mathbf{B}_1 & \mathbf{B}_2 & \cdots & \mathbf{B}_{m-1} & \mathbf{0} \\ \mathbf{u}_0 & \mathbf{u}_1 & \mathbf{u}_2 & \cdots & \mathbf{u}_{m-1} & \lambda \end{bmatrix} \begin{bmatrix} \Delta \boldsymbol{x}_0 \\ \Delta \boldsymbol{x}_1 \\ \Delta \boldsymbol{x}_2 \\ \vdots \\ \Delta \boldsymbol{x}_{m-1} \\ \Delta T \end{bmatrix} = - \begin{bmatrix} \boldsymbol{\psi}_0 \\ \boldsymbol{\psi}_1 \\ \boldsymbol{\psi}_2 \\ \vdots \\ \boldsymbol{\psi}_{m-1} \\ h(\boldsymbol{X}(0)) \\ 0 \end{bmatrix} \tag{5}$$

where $\mathbf{M}_i$ is the i-th monodromy matrix resulting from the derivative of each $\boldsymbol{\psi}_i$ function with respect to $\boldsymbol{x}_i$, and is given by

$$\mathbf{M}_i = \frac{\partial \phi(\boldsymbol{x}_i, \Delta s_i T)}{\partial \boldsymbol{x}_i} \tag{6}$$

Following the pseudo-arclength continuation method, the phase condition and orthogonality condition are also added to Eq. (5). The linear system of equations is solved iteratively in the Newton scheme to obtain corrections to the initial conditions

In order to validate our multiple shooting formulation and methodology, we apply the algorithm to compute NNMs of the two degree of freedom system with cubic stiffness. The system is depicted in Figure 2, and its equation of motion is

$$\begin{aligned} \ddot{x}_1 + (2x_1 - x_2) + 0.5x_1^3 = 0 \\ \ddot{x}_2 + (2x_2 - x_1) = 0 \end{aligned} \tag{7}$$

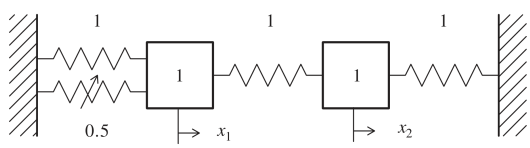


Fig. 2 Schematic illustration of the 2 DoF system [10].

The system above exhibits two NNMs, representing in-phase and out-of-phase motions. The symmetric periodic motion, referred to as $S11+$ in [10], denotes the in-phase motion, and reveals internal resonance branches at higher energy levels. The resonance tongues emanate from the $S11+$ branch and coalesce into the out-of-phase solution branch, demonstrating high-frequency ratio multiples (2:1, 3:1, 4:1, etc.) between the motions.

The FEP for the in-phase branch was computed using the multiple shooting method described, and is shown in Figure 3. Three partitions were chosen, and kept fixed through the continuation process. Zero initial velocity was enforced on both degrees of freedom as the phase condition, and therefore the symmetric NNMs were targeted and captured[1]. Excellent agreement is observed between the multiple shooting and single shooting results through a range of modal interactions, thus validating the formulation of the multiple shooting problem.

Adaptation of the Multiple Shooting Method to Lie Group Formulations

Before the multiple shooting method can be applied to beams modelled on $SE(3)$, it has to be adapted to fit the underlying Lie group framework of the structural model.

We begin by modifying the periodicity condition in Eq. (4). This is due to the nonlinearity of the configuration space of the Lie group manifold, which requires the use of the logarithm map on $SE(3)$ to compute the relative difference between beam configurations. In contrast to the single shooting approach, we need to apply this periodicity condition across all partitions of the multiple shooting method. This results in a Lie group-specific multiple shooting periodicity condition,

[1]These are referred to as "open curve" solutions in the configuration space [10].

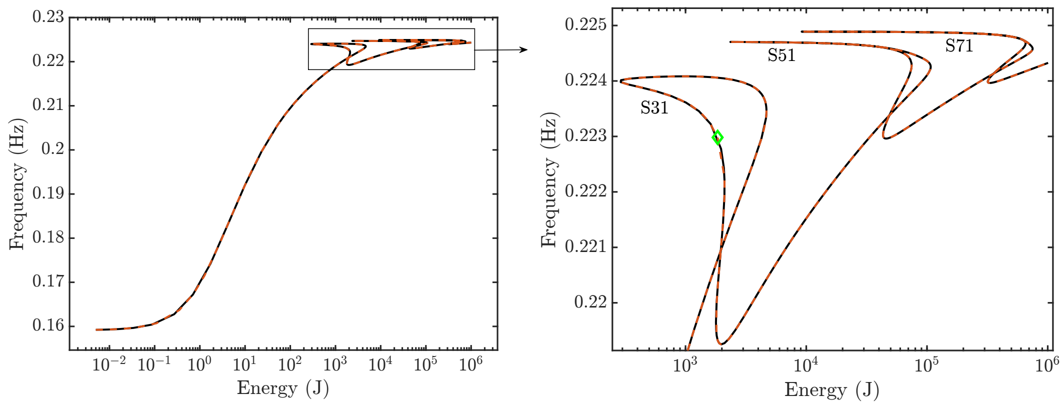


Fig. 3 First NNM of cubic spring. Comparison between single shooting (solid black) and multiple shooting (dashed orange). Symmetric internal resonances with 3:1, 5:1, and 7:1 frequency ratios captured and labelled.

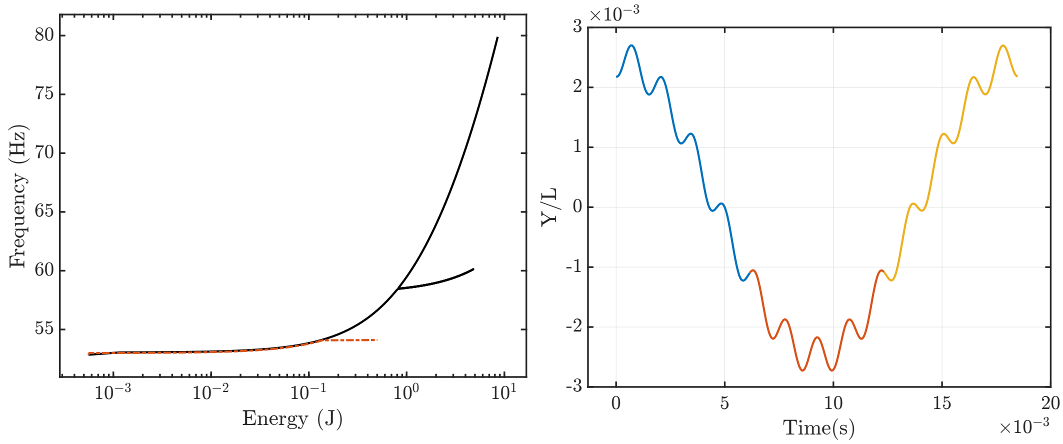
where the beam frames and velocities are compared at the boundaries of each partition to compute the periodicity error. The multiple shooting periodicity function is therefore written as

$$\begin{cases} \boldsymbol{\psi}_0 = \begin{bmatrix} \psi_{\mathbf{H}_0} \\ \psi_{\mathbf{v}_0} \end{bmatrix} = \begin{bmatrix} \log_{SE(3)}(\mathbf{H}_1^{-1}(0)\mathbf{H}_0(\Delta s_1 T))^{\vee} \\ -\mathbf{T}_{SE(3)}^{-1}(-\psi_{\mathbf{H}_0})\mathbf{v}_1(0) + \mathbf{T}_{SE(3)}^{-1}(\psi_{\mathbf{H}_0})\mathbf{v}_0(\Delta s_1 T) \end{bmatrix}, \\ \boldsymbol{\psi}_1 = \begin{bmatrix} \psi_{\mathbf{H}_1} \\ \psi_{\mathbf{v}_1} \end{bmatrix} = \begin{bmatrix} \log_{SE(3)}(\mathbf{H}_2^{-1}(0)\mathbf{H}_1(\Delta s_2 T))^{\vee} \\ -\mathbf{T}_{SE(3)}^{-1}(-\psi_{\mathbf{H}_1})\mathbf{v}_2(0) + \mathbf{T}_{SE(3)}^{-1}(\psi_{\mathbf{H}_1})\mathbf{v}_1(\Delta s_2 T) \end{bmatrix}, \\ \vdots \\ \boldsymbol{\psi}_{m-2} = \begin{bmatrix} \psi_{\mathbf{H}_{m-2}} \\ \psi_{\mathbf{v}_{m-2}} \end{bmatrix} = \begin{bmatrix} \log_{SE(3)}(\mathbf{H}_{m-1}^{-1}(0)\mathbf{H}_{m-2}(\Delta s_{m-1} T))^{\vee} \\ -\mathbf{T}_{SE(3)}^{-1}(-\psi_{\mathbf{H}_{m-2}})\mathbf{v}_{m-1}(0) + \mathbf{T}_{SE(3)}^{-1}(\psi_{\mathbf{H}_{m-2}})\mathbf{v}_{m-2}(\Delta s_{m-1} T) \end{bmatrix}, \\ \boldsymbol{\psi}_{m-1} = \begin{bmatrix} \psi_{\mathbf{H}_{m-1}} \\ \psi_{\mathbf{v}_{m-1}} \end{bmatrix} = \begin{bmatrix} \log_{SE(3)}(\mathbf{H}_0^{-1}(0)\mathbf{H}_{m-1}(\Delta s_m T))^{\vee} \\ -\mathbf{T}_{SE(3)}^{-1}(-\psi_{\mathbf{H}_{m-1}})\mathbf{v}_0(0) + \mathbf{T}_{SE(3)}^{-1}(\psi_{\mathbf{H}_{m-1}})\mathbf{v}_{m-1}(\Delta s_m T) \end{bmatrix} \end{cases} \tag{8}$$

where $\mathbf{H}_i(0) \in SE(3)$ and $\mathbf{v}_i(0) \in \mathfrak{se}(3)$ are, respectively, the initial frames and velocities for each partition. The final frames and velocities at the end of the time integration along each partition, are denoted by $\mathbf{H}_i(\Delta s_i T)$ and $\mathbf{v}_i(\Delta s_i T)$ respectively. The $2N \times 1$ vector of initial conditions for each partition will now consist of the N initial increments and N initial velocities, that is $\boldsymbol{x}_i = [\mathbf{h}_i(0) \;\; \mathbf{v}_i(0)]^T$, where the increments give the initial beam configuration through the application of the exponential map and N is the number of degrees of freedom of the model.

Application of Multiple Shooting to 2D Beams

A 2D straight clamped-clamped beam is considered and its undamped periodic solutions (Nonlinear Normal Modes) are computed with multiple shooting and pseudo-arclength continuation. The beam is discretised with 30 elements, and its geometric and material properties are given as: $E = 210\,GPa$, $\rho = 7850\,kg/m^3$, $L = 1\,m$, $A = 10^{-4}\,m^2$. The beam is modelled using the SE Lie group framework in two dimensions [11], namely the $SE(2)$ group, which is the group of rigid body transformations in two dimensions. The frequency energy plot of the first NNM of the beam is shown in Figure 4a which depicts the periodic solution branches computed with single shooting and multiple shooting with 3 partitions. It is seen that the multiple shooting branch captures modal interactions of higher order, evident by the branching of the interaction tongue from the main backbone curve. The time series of a solution on this tongue is depicted in Figure 4b, showing the periodic motion of the beam midpoint across all 3 segments of the multiple shooting partitions. The multiple shooting method is seen to accurately capture the periodic motion of the beam. However, the robustness of the current implementation needs to be improved to ensure convergence along the full range of the backbone curve at higher energy levels, and to avoid high order branching which is a common issue in the continuation of periodic solutions using shooting.



(a) Frequency Energy Plot.

(b) Time Series.

Fig. 4 First NNM of doubly clamped beam. Comparison between single shooting (solid black) and multiple shooting (dashed orange). Time series of a multiple shooting solution on the interaction tongue shown.

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