Chapter 17
Forced Response of Nonlinear Systems Under Multi-Frequency Excitations by Using Harmonic Balance Method

Dersu Celiksoz and Ender Cigeroglu

Abstract Even though most of the engineering systems falls under the category of single-frequency excitation, there exist several systems, such as double rotor systems, turbomachinery airfoils, gear boxes, micromotors etc., where the excitation is composed of multiple frequencies. Therefore, this study explores the influence of multi-frequency excitation on the vibration behavior of a nonlinear dynamic system. A system with gap nonlinearity is examined in order to better understand the behavior of the discontinuous nonlinear elements under harmonic forces that are exhibiting different frequency characteristics. The multi-frequency excitation is modeled to include two frequencies that are non-integer multiples of a fundamental frequency. Using harmonic balance method, parametrical modal domain analysis is conducted to present the behavior of the system when multiple excitation frequencies are either close to each other or distinct. The findings also illustrate the relationships between excitation sources and the resulting vibration responses of each multi-frequency harmonics. Based on the observed relations, contributions of the harmonics are evaluated and prediction of the vibration sources within a specific frequency range is facilitated.

Keywords Multi-Frequency Excitations · Nonlinear Dynamic Systems · Nonlinear Vibrations · Gap Nonlinearity · Harmonic Balance Method

Introduction

In dynamic systems, mechanical vibrations caused by external sources are considered during the design, analysis, and maintenance phases to ensure the stability and reliability of operating components. Various excitation sources exist for such systems, most of which oscillate at a single frequency. However, some engineering systems, particularly those operating in complex environments, are excited by forces with multiple frequencies. Examples of such systems include double rotor systems, turbomachinery airfoils, gearboxes, and micromotors. Understanding the dynamics and vibration behavior of engineering systems excited by multiple frequencies is crucial for predicting loading conditions and the response of mechanical components.

Unlike single-frequency excited engineering problems, multi-frequency cases require special attention to solve the forced response of the systems, since the harmonics are not integer multiples of each other. Moreover, the effect of multi-frequency phenomena becomes more complicated in the presence of nonlinear elements that influence the frequency response significantly. One of the most common types of nonlinear elements in rotating mechanical systems is gap nonlinearity. Gap nonlinearity arises in systems with contact separations or clearances where mechanical connections or joints exist. These systems exhibit response-dependent system stiffness, since some connections only occur if certain response conditions are met. In other words, the stiffness varies based on the system response, such as displacements or load thresholds are exceeded. This leads to nonlinear behavior, and modeling such systems requires attention to these threshold values. Several studies from different application areas have been conducted on gap nonlinearity in mechanical elements. For instance, Okuizumi [1] investigated the influence of gap size on the nonlinear vibration characteristics of a satellite truss structure with a gap element. Wang et al. [2] examined a non-direct-drive servo motor system with gap nonlinearity to propose a strategy for improving system robustness. Based on such findings, it can be concluded that further investigation of gap nonlinearity across various discontinuous mechanical systems is crucial.

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For the periodic problems of differential algebraic equations, Harmonic Balance Method (HBM) is used to find a numerical solution. As noted by Jazar [3], substituting an assumed periodic solution using Fourier series into the differential equation generates a residual term. The method requires that the Fourier coefficients of this residual term should vanish, ensuring a balanced solution. Additionally, variants and extensions of the HBM have been applied to diverse types of problems as summarized in [4].

When subjected to multi-frequency excitations, nonlinear systems produce responses whose dynamics are difficult to predict using conventional methods. Thus, different approaches have been studied for such excitation cases. Junge et al. [5] proposed a creative harmonic balance technique based on multidimensional Fourier transformations. This technique extends the standard harmonic balance method by constructing separate time domains for each base frequency. Furthermore, Prabith and Krishna [6] developed a time variational method (TVM) for analyzing mechanical systems experiencing multiple-frequency excitations. This analysis is simplified compared to the multi-harmonic balance method and avoids the need to transform between frequency and time domains. Wu et al. [7] refined the semi-analytical HBM variants by proposing a new formulation of the Alternating Frequency-Time method and introduced a new arc-length continuation condition to expand the capabilities for solving quasi-periodic excitations. Guskov and Thouverez [8] developed a multidimensional HBM and an adjusted HBM to address the nonlinear response and stability of a duffing oscillator subject to bi-periodic excitation. Also, Huang and Zhu [9] proposed a novel incremental harmonic balance method that employs two timescales to handle quasiperiodic motion in nonlinear systems, allowing precise identification of quasiperiodic behavior.

Considering the findings of prior research, this study focuses on the influence of multi-frequency excitation on the vibration behavior of a nonlinear dynamic system with a gap element. Excitation forces are modeled by considering two excitation frequencies that are non-integer multiples of a fundamental frequency. This type of forcing represents a common scenario in rotating systems with two excitation sources. To solve the equation of motion, HBM is extended to balance the harmonic components including different combinations of multiple excitation frequencies and to cover contribution of each harmonic term. The study also examines system response when the excitation frequencies are either close to each other or relatively distinct. Overall, this research provides valuable insights into the design and analysis of dynamic systems subjected to multi-frequency excitations.

Mathematical Modeling

As shown in Fig. 1, single degree of freedom nonlinear system with gap nonlinearity under the action of multi-frequency excitation is considered in this study. The quasi-periodic external forcing and system response are assumed to be functions of two excitation frequencies which are non-integer multiples of a fundamental frequency.

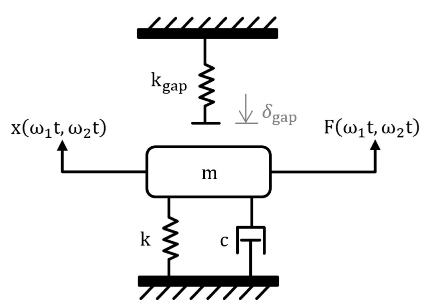

Fig. 1 Nonlinear dynamic system model with gap element.

The equation of motion of the nonlinear system can be written as follows.

$$m\ddot{x}\,(t) + c\dot{x}\,(t) + kx\,(t) + k_{gap}\,(x\,(t) - \delta_{gap})\,g\,(x) = F\,(t)\,, \tag{1}$$

$$g\,(x) = \left\{ \begin{array}{ll} 1, & x \geq \delta_{gap} \\ 0, & x < \delta_{gap} \end{array} \right. . \tag{2}$$

where, x is the displacement of the system. m, k, c and F are the mass, stiffness, viscous damping in the system and external forcing acting on the system, respectively. k_{gap} and δ_{gap} are the gap stiffness and gap value of the gap nonlinearity. Furthermore, $g\,(x)$ is used to define when the gap is closed and stiffness is in contact with the system. Plot of the nonlinear force for the gap element is shown in Fig. 2.

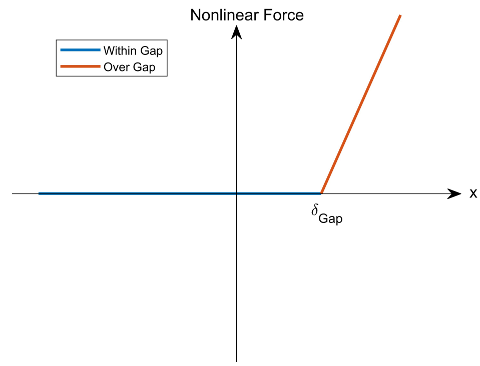

Fig. 2 Variation of nonlinear force.

The quasi-periodic response and forcing are assumed to be functions of two excitation frequencies. Similar to the multi-dimensional representation described by Guskov [8], the response and forcing are defined in Eqs. (3) and (4) by considering the contribution of different harmonic variations of the excitation frequencies.

$$x\left(t\right)=x_0+\sum_{(k_1,k_2)\in \mathrm{Z}^2}\left[x_{s(k_1,k_2)}\sin\left(k_1\omega_1 t+k_2\omega_2 t\right)\ +x_{c(k_1,k_2)}\cos\left(k_1\omega_1 t+k_2\omega_2 t\right)\right], \tag{3}$$

$$F\left(t\right)=f_0+\sum_{(k_1,k_2)\in \mathrm{Z}^2}\left[f_{s(k_1,k_2)}\sin\left(k_1\omega_1 t+k_2\omega_2 t\right)\ +f_{c(k_1,k_2)}\cos\left(k_1\omega_1 t+k_2\omega_2 t\right)\right]. \tag{4}$$

Note that, k_1 and k_2 in the equations represent harmonic multipliers within the integer-couple domain Z^2, which includes various combinations of multipliers up to a specified harmonic number. Also, x_0, $x_{s(k_1,k_2)}$, $x_{c(k_1,k_2)}$, f_0, $f_{s(k_1,k_2)}$ and $f_{c(k_1,k_2)}$ are used for the Fourier coefficients of corresponding terms.

Furthermore, the excitation frequencies are defined in the Eq. (5) as non-integer multiples of a fundamental frequency of ω_f, with the multipliers of β_1 and β_2.

$$\omega_1=\beta_1\omega_f,\ \omega_2=\beta_2\omega_f. \tag{5}$$

Harmonic Balance Method For Multi-Frequency Excitation

This work analyzes the system response in the frequency domain using an enhanced harmonic balance method. Following the summary by Sert and Cigeroglu [10], the Multi-Harmonic Balance Method is utilized to address systems defined by a finite number of harmonics. The residual equation, including selected harmonics, is derived by combining both linear and nonlinear terms into a matrix form for systems excited by single-frequency as follows.

$$\boldsymbol{r}\left(\omega\right)=\begin{bmatrix} k & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{\Lambda}\left(\omega\right) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}\left(2\omega\right) & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{\Lambda}\left(n_H\omega\right) \end{bmatrix}\begin{Bmatrix} x_0 \\ x_{s1} \\ x_{c1} \\ x_{s2} \\ x_{c2} \\ \dots \\ x_{sn_H} \\ x_{cn_H} \end{Bmatrix}+\begin{Bmatrix} f_{N0} \\ f_{Ns1} \\ f_{Nc1} \\ f_{Ns2} \\ f_{Nc2} \\ \dots \\ f_{Nsn_H} \\ f_{NcN_H} \end{Bmatrix}-\begin{Bmatrix} F_0 \\ F_{s1} \\ F_{c1} \\ F_{s2} \\ F_{c2} \\ \dots \\ F_{sn_H} \\ F_{cN_H} \end{Bmatrix}=0. \tag{6}$$

In Eq. (6), $\mathbf{\Lambda}(\omega)$ represents the dynamic stiffness of each harmonic up to the harmonic number of n_H, and defines the linear portion of the residual as follows.

$$\mathbf{\Lambda}\left(\omega\right)=\begin{bmatrix} k-\omega^2 m & -\omega c \\ \omega c & k-\omega^2 m \end{bmatrix}. \tag{7}$$

Moreover, f_{N_r} are used to denote the bias, sine or cosine components of the nonlinear internal forcing. Building upon these studies on the Multi-Harmonic Balance Method, as well as the assumed solutions and force representations provided by Eqs. (3) and (4), the HBM is extended to address a nonlinear problem under multi-frequency excitation. As a result, the modified residual equation and corresponding dynamic stiffness are rederived as follows.

$$\boldsymbol{r}(\omega_1,\omega_2) = \begin{bmatrix} k & \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}(\omega_1,0) & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}(0,\omega_2) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}(\omega_1,\omega_2) & \dots & \mathbf{0} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \boldsymbol{\Lambda}(n_{H1}\omega_1, n_{H2}\omega_2) \end{bmatrix} \begin{Bmatrix} x_{0(0,0)} \\ x_{s(1,0)} \\ x_{c(1,0)} \\ x_{s(0,1)} \\ x_{c(0,1)} \\ x_{s(1,1)} \\ x_{c(1,1)} \\ \vdots \\ x_{s(n_{H1},n_{H2})} \\ x_{c(n_{H1},n_{H2})} \end{Bmatrix} +$$

$$\begin{Bmatrix} f_{N(0,0)} \\ f_{Ns(1,0)} \\ f_{Nc(1,0)} \\ f_{Ns(0,1)} \\ f_{Nc(0,1)} \\ f_{Ns(1,1)} \\ f_{Nc(1,1)} \\ \vdots \\ f_{Ns(n_{H1},n_{H2})} \\ f_{Nc(n_{H1},n_{H2})} \end{Bmatrix} - \begin{Bmatrix} F_{(0,0)} \\ F_{s(1,0)} \\ F_{c(1,0)} \\ F_{s(0,1)} \\ F_{c(0,1)} \\ F_{s(1,1)} \\ F_{c(1,1)} \\ \vdots \\ F_{s(n_{H1},n_{H2})} \\ F_{c(n_{H1},n_{H2})} \end{Bmatrix} = 0 \tag{8}$$

$$\boldsymbol{\Lambda}(\omega_1,\omega_2) = \begin{bmatrix} k-(\omega_1+\omega_2)^2 m & -(\omega_1+\omega_2)c \\ (\omega_1+\omega_2)c & k-(\omega_1+\omega_2)^2 m \end{bmatrix} \tag{9}$$

Note that the terms of n_{H1} and n_{H2} in Eqs. (8) and (9) represents the number of considered harmonics for each excitation frequency. Additionally, the nonlinear force resulted from nonlinear gap element is denoted by $f_{N(k_1,k_2)}$ and it is calculated using Fourier integrals for each harmonic combination. This integration involves calculating the nonlinear force contributions over a common period to capture the nonlinear behavior across the multi-frequency range in which the gap threshold is exceeded.

Following the selection of the contributing harmonics, the residual equation is solved using the arc length continuation method, which is effective in handling nonlinearities through a path-following approach. This solution process involves predicting a future solution that lies on a circle being centered on the current solution and having a predefined radius in the solution domain. The predicted solution is then iteratively corrected employing the Newton-Raphson method until the convergence threshold is met. Details of Newton's method with arc-length continuation can be found in [11–12].

Results and Discussion

The analysis approach described in the previous section is conducted for various cases where different multipliers are used for the excitation frequencies. This parametric study aims to understand the behavior of the system when multiple excitation frequencies are either close to each other or distinct, as well as the influence of harmonic pairs on the solution. First, the harmonic pairs present in the system are specified and their multipliers used in the study are given in Table 1. The quasi-periodic response of the system is obtained by substitution of those harmonic pairs into the multi-frequency response given by Eq. (3). Furthermore, external forcing is selected to excite the system with two different frequencies, as shown in Eq. (10).

$$F(t) = f_{s(1,0)}\sin(\omega_1 t) + f_{s(0,1)}\sin\left(\omega_2 t + \phi_{(0,1)}\right). \tag{10}$$

Table 1 Multipliers of selected harmonic couples

Harmonic Couple	1	2	3	4	5	6
$\mathbf{k_1}$	0	1	1	1	0	-1
$\mathbf{k_2}$	0	0	1	-1	1	1

Subsequently, a system model and corresponding residual equation are created using the parameters provided in Table 2. The frequency response of the system is obtained employing Newton's Method with the arc length continuation method. For clarity, phase angle is assumed to be 0 in the analysis and its effect is investigated separately.

Table 2 System Parameters.

Parameter	Unit	Value
Mass, $\mathbf{m}$	kg	2.00
Stiffness, $\mathbf{k}$	N/m	300
Damping, $\mathbf{c}$	Ns/m	0.30
Gap Stiffness, $\mathbf{k_{gap}}$	N/m	200
Gap Distance, $\delta_{\mathbf{gap}}$	m	0.03
External Forcing, $\mathbf{f_{s(1,0)}}$	N	0.50
External Forcing, $\mathbf{f_{s(0,1)}}$	N	0.35
Phase Angle, $\phi_{(\mathbf{0,1})}$	rad	0

To better understand the contributions of multi-frequency harmonics, the frequency response results for non-integer frequency multipliers of 0.75 and 1.20, as defined in Eq. (5), are obtained, and shown in Figs. 3 and 4 both. The results include the total solution along with the harmonic solution components. The linear response under multi-frequency excitation is also included in the figures in order to highlight the influence of nonlinearity.

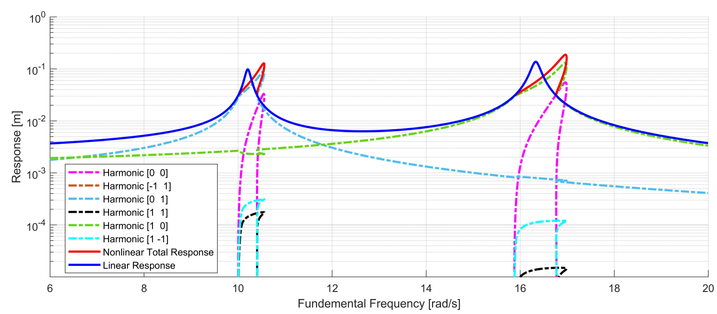

Fig. 3 Frequency response analysis results in logarithmic scale for $\beta_1 = 0.75$ and $\beta_2 = 1.20$ within fundamental frequency range.

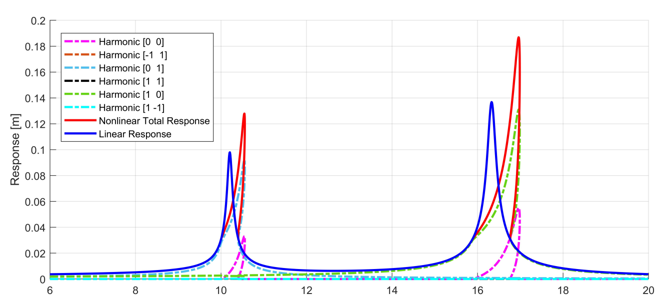

Fig. 4 Frequency response analysis results in linear scale for $\beta_1 = 0.75$ and $\beta_2 = 1.20$ within fundamental frequency range.

By investigating the resonance frequencies given in Figs. 3 and 4, it can be observed that resonance occurs at the specified non-integer multiples of the linear natural frequency. These resonance points are primarily due to the contributions of the fundamental harmonic of each excitation component. For instance, the first resonance occurs around 10.20 rad/s, which, when multiplied by the harmonic multiplier of 1.2, results in a natural frequency of 12.25 rad/s. Bias term is smaller than the fundamental harmonics but it is significantly larger than the combination harmonics considered in the study.

The analysis discussed above is repeated for different non-integer multiples of the fundamental frequency to examine the effect of the closeness of the multipliers. A range of multiplier pairs, starting from the same excitation frequency to excitation frequencies away from each other, are considered and given in Table 3.

Table 3 Non-integer multiplier pairs for the parametrical analysis.

Multiplier Pair	1	2	3	4
β_1	0.75	0.75	0.75	0.75
β_2	0.75	0.80	1.2	1.45

For each set of multipliers, the analysis is conducted, and the total responses of each case are given as a single plot, as shown in Fig. 5. The results indicate that as the multipliers become closer, the maximum response amplitude increases. This is because the close resonances of the corresponding harmonics result in more interaction, leading to a higher overall response. Conversely, when the multipliers are apart from each other, the distinct harmonics behave more independently and reduce their interaction, which leads to smaller response amplitudes. This is an expected result, since in the case of distant natural frequencies, energy given by the excitation forces are applied at different frequencies: whereas, for close frequencies energy of the excitations are localized in a narrow frequency range resulting in the increase of the response amplitude.

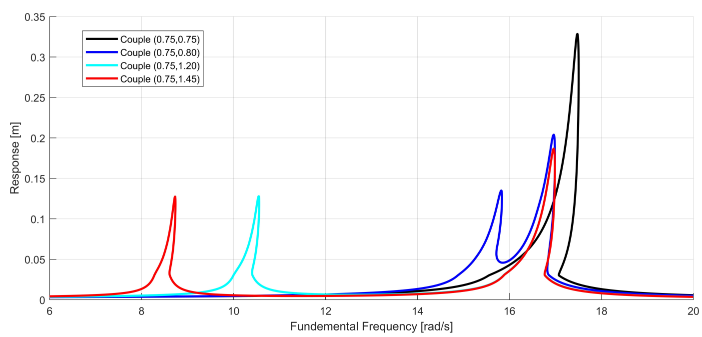

Fig. 5 Total frequency responses for selected multiplier pairs within fundamental frequency range.

In addition to the study of the effect of multiplier pairs on the response, the influence of the phase angle between harmonic components of the forcing is investigated. By keeping the multiplier pairs of 0.75 and 1.20 constant, analysis is repeated for different phase angles of 0, $\pi/3$ and $4\pi/3$ radians. It is observed that results are affected from the phase angle variation negligibly; hence, for brevity these results are not given here.

Conclusion

This research emphasizes the impact of multi-frequency excitation on the vibrational behavior of nonlinear dynamic systems by examining a mechanical system with gap nonlinearity. The study provides important findings by modeling the system using an extended harmonic balance method for multi-frequency excitation cases and solving the forced response with the arc-length continuation method. Firstly, it investigates the harmonic contributions that are non-integer multiples of the fundamental frequency. Secondly, it conducts a parametric study to understand the system's behavior as the frequency multiples get closer. The results indicate that as the excitation frequencies come closer together, the resultant vibrations become stronger and require careful design considerations for these systems. Furthermore, the research highlights the potential for identifying unexpected vibration sources by analyzing the system's multi-frequency response. In conclusion, understanding multi-frequency excitations and their effects is crucial for enhancing the reliability and performance of mechanical systems.

Future research extending this analysis to more complex mechanical systems could lead to a better understanding of various engineering systems under multi-frequency excitations.

References

1. Okuizumi, N. and Natoriy, M.C., "Nonlinear Vibrations of a Satellite Truss Structure with Gaps" *45th AIAA/ASME/ASCE/AHS/ASC Structures, Structural Dynamics & Materials Conference*, pp. 19–22, 2004.
2. Wang, B., Ji, R., Zhou, C., Sohel, R.M., Liu, K., Hua, W., and Ye, H., "A Gap Nonlinearity Compensation Strategy for Non-Direct-Drive Servo Systems" *Energies*, vol. 17, p. 3582, 2024.
3. Jazar, R.N., Perturbation Methods in Science and Engineering, *Springer*, pp. 262–277, 2021.
4. Yan, Z., Dai, H., Wang, Q., and Atluri, S.N., "Harmonic Balance Methods: A Review and Recent Developments" *Computer Modeling in Engineering & Sciences*, vol. 137(2), pp. 1419–1459, 2023.
5. Junge, L., Frey, C., Ashcroft, G., and Kugeler, E., "A New Harmonic Balance Approach Using Multidimensional Time" *Journal of Engineering for Gas Turbines and Power*, vol. 143, 2021.
6. Prabith, K. and Krishna, I.R.P., "A Time Variational Method for the Approximate Solution of Nonlinear Systems Undergoing Multiple-Frequency Excitations" *Journal of Computational and Nonlinear Dynamics*, vol. 15(3) 2020.
7. Wu, J., Hong, L., and Jiang, J., "A Comparative Study on Multi- and Variable-Coefficient Harmonic Balance Methods for Quasi-Periodic Solutions" *Mechanical Systems and Signal Processing*, vol. 187, 2023.
8. Guskov, M. and Thouverez, F., "Harmonic Balance-Based Approach for Quasi-Periodic Motions and Stability Analysis" *Journal of Vibration and Acoustics*, vol. 134, pp. 031003-1–031003-11, 2012.
9. Huang, J.L. and Zhu, W.D., "An Incremental Harmonic Balance Method with Two Timescales for Quasiperiodic Motion of Nonlinear Systems Whose Spectrum Contains Uniformly Spaced Sideband Frequencies" *Nonlinear Dyn*, vol. 90(2), pp. 1015–1033, 2017.
10. Sert, O. and Cigeroglu, E., "Adaptive Harmonic Balance Methods, A Comparison" *Special Topics in Structural Dynamics: Conference Proceedings of the Society for Experimental Mechanics Series, The Society for Experimental Mechanics*, vol. 6, pp. 279–289, 2016.
11. Von Groll, G., and Ewins, D. J., "The Harmonic Balance Method with Arc-Length Continuation in Rotor/Stator Contact Problems" *Journal of Sound and Vibration*, vol. 241(2), pp. 223–233, 2001.
12. Dimitriadis, G., "Continuation of higher-order harmonic balance solutions for nonlinear aeroelastic systems" *Journal of Aircraft*, vol. 45(2), pp. 523–537, 2008.