Chapter 3
Nonlinear Frequency Response Surfaces, Part I: Theoretical Formulation

D. Di Maio, Jip van Tiggelen, and Marcel H.M. Ellenbroek

Abstract The Frequency Response Functions (FRF) are the most widely used functions to characterize the dynamic behaviour of structures. The natural frequencies and damping behaviour can be easily and quickly detected from a Bode diagram. FRFs are used to calculate the modal properties by modal analysis methods, and in the last step, a frequency response model is used to synthesize the response functions to verify the quality of the identified modal properties identified. The circularity between 1) measurement, 2) identification, 3) regeneration and 4) comparison is ensured on the assumption that transfer functions are measured under strict conditions, such as linear vibrations. All mechanical systems are intrinsically nonlinear. Some sources of nonlinearity might be excited and revealed, and others not for various reasons. Anyhow, it is unavoidable to measure nonlinear vibrations when vibration tests are executed at various levels of excitation forces, and both linear and nonlinear vibrations are processed to obtain linear and nonlinear FRFs. The linear FRFs can be processed using the existing linear identification methods. The nonlinear FRFs are either stored in archive, or blandly processed to evaluate the level and the type of nonlinearity, such as hardening or softening behaviour.

This research aims to (i) formulate a new frequency response model and (ii) formulate a new identification method for extracting amplitude-dependent modal parameters. The first objective will demonstrate that a nonlinear frequency response function lays on a Force Plane which intercepts an FRF surface generated by equivalent linear FRFs. The second objective will demonstrate that a linear modal analysis method, called line-fit and based on the Dobson formulation, allows extracting amplitude-dependent modal parameters from nonlinear FRFs. By combining the results achieved by the two research objectives, one can regenerate and compare measured and synthetic nonlinear transfer functions.

Keywords Forced vibrations · system assembly · the nonlinearity of joints · Nonlinear Frequency Response Surface

Introduction

In the aerospace and space industry, vibration qualification tests serve an essential purpose - to evaluate the behaviour of a structure in conditions that it would typically experience during launch or flight. However, the current approach of performing qualification tests on fully assembled systems and sub-systems has its limitations. By the time the dynamic behaviour is tested, it may be too late to make any design changes that would have been required, which can result in costly redesigns and delays. Thus, it is crucial to develop a more proactive approach to assessing the vibration response of systems earlier on during the design and make.

While modelling the individual components in a system can be relatively straightforward, accurately modelling their assembly, particularly the connections between the components, is complicated due to the nonlinear transfer behaviour at the joints between components [1]. As such, predictive modelling of the forced vibrations of assembled structures has been challenging for many years [2, 3]. Reducing the computational efforts required for solving the dynamics response of nonlinear systems by lumped parameter joint models, such as the Iwan model [4, 5] or assuming a polynomial function. These could be considered phenomenological models and not fundamental models describing the physics, thus still requiring expert

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knowledge or calibration using test data. On the other hand, more efficient solution methods were developed for obtaining the forced vibration behaviour of nonlinear systems (as compared to i.e. direct time integration), such as harmonic balance method [6], continuation-based methods [7] and quasi-static approaches [8, 9]. Yet, prediction of the nonlinear dynamic behaviour of joints remains a challenge (in an industrial context).

This short context will help the reader understand that one of the main shortfalls can be seen in the characterization of the vibration response, which becomes nonlinear for several reasons, where interfaces play a large part in such a challenge. Although academia has developed several testing and analysis techniques for characterizing nonlinear vibrations during the past years, industry integrated a tiny fraction of that knowledge [10]. The main reason is the implementation of the proposed methods into industrial standards, which still relies on the measurement of Frequency Response Functions (FRF) for characterizing the dynamics of structures. Engineers find FRFs very useful because they can relate the response at any point on the structure to the input force exerted elsewhere. This means that an engineer can evaluate an appropriate response to design requirements. Unfortunately, the nonlinear vibrations disrupt the established procedures. Therefore, this paper aims to offer an approach that is simple and easy to integrate into the industry standards, thus providing invaluable information to engineers. This paper is divided into two parts, one more theoretically focussed and one more application-driven. The latter was authored by Jip van Tigglen and is listed in the same conference proceedings.

Theoretical Formulations of the Nonlinear Frequency Response Surface

This section will introduce the theoretical formulation of a nonlinear frequency response surface. It is a major breakthrough for engineering science and the practice of experimental modal testing. It is a custom to perform modal testing on structures before any other vibration test is executed. It is important to identify the natural frequencies, damping values, and mode shapes, all three of which form the modal parameters that can be used to characterize a structure's dynamic. Furthermore, these modal parameters validate the experimental model of the mechanical system. It is common practice to perform the modal testing using any desired excitation force and process the test data with modal analysis tools. The final verification of the data analysis consists of synthesizing the FRFs and overlaying them on the experimental ones to check if the modal analysis evaluated the modal parameters correctly. The process described so far stops when a test engineer acquires nonlinear test data, meaning that FRFs are distorted because of sources of nonlinearity. Hence, the modal analysis stops working correctly. The following sections will show how carrying out the same modal identification process is possible but using a different analysis method. Most of the research is already available in [11], so this paper offers a concise summary explaining the fundamental steps.

Linear frequency response surface (LFRS)

The relationship between force and displacement for a single degree of freedom system is linear when there is a single coefficient describing the system's stiffness. Many engineers are accustomed to seeing this relationship made a constant stiffness line, which intercepts the mass line and the system's natural frequency. The frequency response is linear for any vibration amplitude. Figure 1 shows a hypothetical waterfall of linear responses forming a Linear Frequency Response Surface. As said the stiffness of the system is not amplitude dependent; therefore, every FRF will start from the same stiffness amplitude. It is also simple to understand that any constant amplitude force plane, created by expanding over the frequency range of the red dashed line, cuts across the LFRS and will produce a linear FRF that lies on that force surface. Although this description reads very clearly, a linear system will produce linear responses, and the concept of frequency response surface will serve to understand the transition from linear to nonlinear vibrations.

Nonlinear Frequency Response Surface (NFRS)

This sub-section will discuss the transition form linear to nonlinear vibrations. It is worth mentioning that, hereafter, the nonlinear vibrations will be assumed to be at a steady state, for which the FRF is the ratio between the input and output at the fundamental drive frequency of the excitation. Hence, any high-order harmonics is neglected. Figure 2 shows the static receptance for a single DOF system whose stiffness relationship is amplitude-dependent. This dependency implies that any FRF will start from a different linear equivalent stiffness. The figure shows the system's linear and nonlinear static receptance, indicating receptance as the ratio between force and displacement.

Having generated a nonlinear static receptance using a cubic stiffness nonlinearity, the following step is to generate the Nonlinear Frequency Response Surface. Therefore, one can still use a linear frequency response model, starting from an equivalent linear stiffness amplitude, which is likewise carried out for the linear frequency response surface. One must catch this idea's major novelty, which suggests that a linear response model is still a valid selection for generating a nonlinear

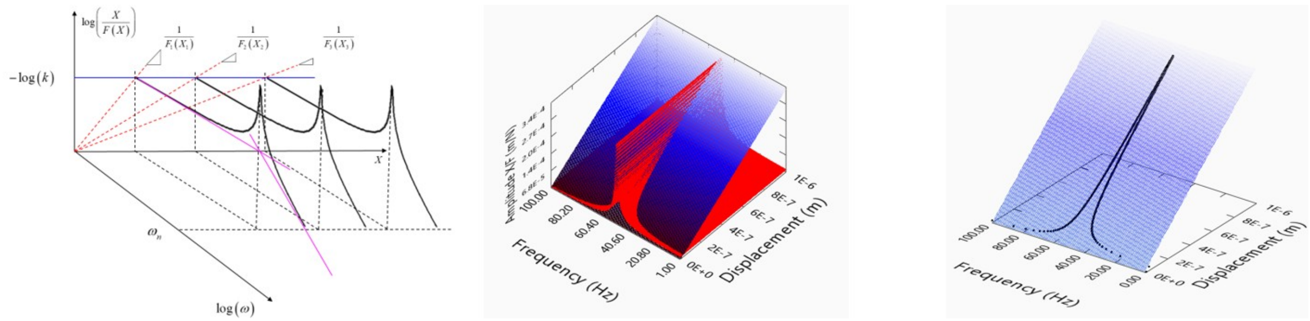

Fig. 1 Linear frequency response surface (LFRS)

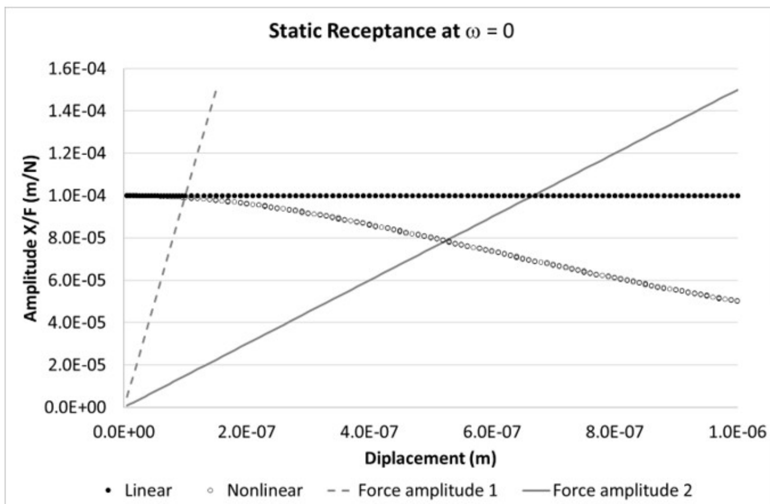

Fig. 2 Linear and nonlinear static receptance

response solution. Figure 3 shows the 3D NFRS in red and the constant amplitude force surface in blue. The nonlinear FRF is the unique solution between the interception of the NFRS and the desired force plane, the inclination of which can be changed. The concluding remarks of this section are straightforward. One can understand that the same NFRS can be generated by accessing modal amplitude-dependent parameters. The response model equation (1) can be used to generate that NFRS once the amplitude-dependent parameters are identified. The identification process will be briefly explained on the following section and the application to an experimental case will be provided as an example.

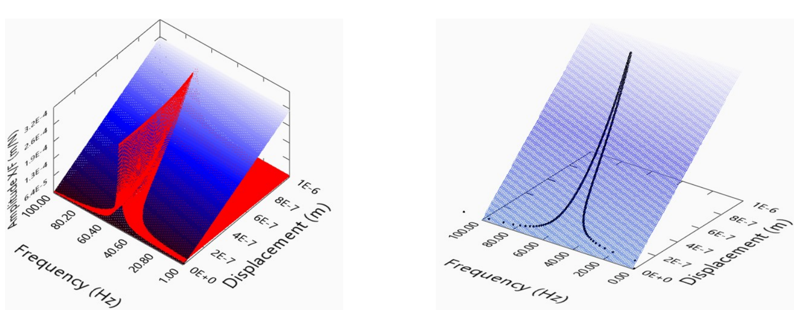

Fig. 3 Nonlinear Frequency Response Surface (NFRS) (red) and constant amplitude force plane (blue)

Generation of the NFRS Using the Experimental Modal Analysis

This section presents an experimental modal analysis technique for evaluating the amplitude-dependent parameters of nonlinear FRFs. As introduced, many test engineers find the data analysis of nonlinear FRFs very challenging because of the few methods available to process FRF data with distortion caused by source of nonlinearity. Furthermore, assuming that modal parameters are evaluated, the synthesis of an FRF is not straightforward, and thus, the assessment of the modal parameters is left unchecked. The previous section addressed the problem of generating nonlinear FRFs, which can be used for synthesizing nonlinear FRFs from the EMA approach.

EMA using the modified-Dobson method

The experimental identification is carried out on FRFs that are measured under steady-state conditions and for reasonably well-separated modes to avoid mode coupling. The Dobson method [11] is a modal analysis technique that allows using one of the properties of the inverse of the FRF. It works by line-fitting the real and imaginary parts of the inverse of the FRF and using the line-fitting coefficients to extract the modal parameters. The method is not used as often for linear modal analysis because of the more powerful techniques developed during the past two decades. However, it proves the best identification method for nonlinear FRF analysis. The underlying principle is the following. Three Nyquist points identify a circle, the properties of which can be curve-fit to identify the dynamics of a linear system. The circle -fit method was one of the most used techniques in the past for this purpose. Unfortunately, the nonlinear vibrations transform the circle into something else, for the stiffness nonlinearity part of the circle disappears, as shown in Figure 4 see the grey hollow dots. The circle stops at a certain amplitude level and then resumes at lower amplitudes. The response amplitude jump causes this because of the stiffness nonlinearity used to generate a numerical nonlinear FRF. The smart analysis, which exploits the Dobson method, fixes two frequency points at the lowest amplitude (the linear one) and uses a third one that sweeps all the other frequency points. Hence, every triplet of frequency points generates an equivalent Nyquist circle or an equivalent linear dynamical system. The goal is to observe how the dynamics of a system evolves from linear to nonlinear vibrations. Hence, one can generate as many triplets as the frequency points measured. By applying the line-fitting to the triplets, one can evaluate the modal properties at that vibration amplitude, thus evaluating an equivalent linear system. By repeating the analysis process, one can observe how the natural frequency curve changes because of the amount of vibrations. The reader is encouraged to expand this quite brief description by reading the full paper [11]. Figure 5 shows the final step of the analysis, where the natural frequency curve is evaluated at every amplitude level of the nonlinear FRF. The black dots of the natural frequency curve indicate the analysis carried out using the frequency points from 45 to approx. 51 Hz, while the blue dots the other part of the FRF branch. Hence, equation (1) can be used for generating an NFRS.

$$\alpha(\omega, X) = \frac{{}_rA(X)}{\omega_r^2(X) - \omega^2 + i\eta_r(X)\omega_r^2(X)}$$

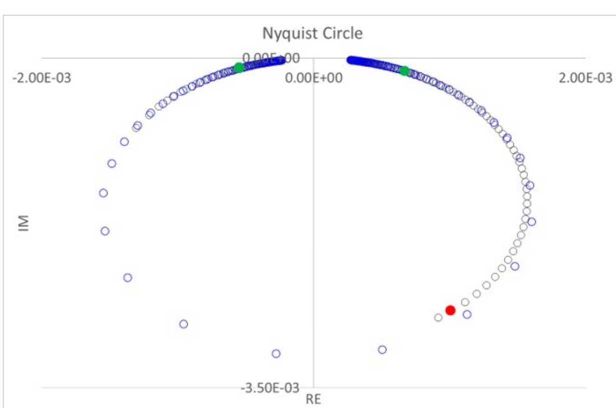

Fig. 4 Nyquist circles for linear (blue hollow dots) and nonlinear FRFs (grey hollow dots)

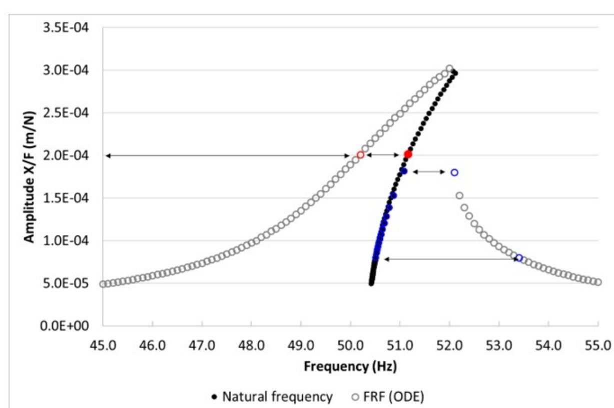

Fig. 5 Nonlinear FRF with the natural frequency curve

Application to an experimental blind case

Section 2 showed that a nonlinear FRF is the unique solution between an NFRS intercepting a force surface of any desired amplitude. The idea was demonstrated using a simple SDOF nonlinear system. However, engineers need to deal with real nonlinear FRFs and be able to evaluate properties that can be used to evaluate the system response at various amplitudes of vibration. This section will show an example of a blind case where the nonlinear FRF was provided without any further information of the test article or type of test. Besides, the FRF was a ratio between the base acceleration and the response measured at a point of the test article. Hence, it is a perfect case study to prove that the approach described so far is valid.

The process consists of:

1. Perform the modified-Dobson method to extract amplitude-dependent modal parameters.
2. Use equation (1) to generate a NFRS.
3. Decide the amplitude(s) of the force plane to extract the synthesize nonlinear FRF.
4. Overlay the measured and synthesized nonlinear FRFs.

Figure 6 shows the measured FRF in red. It was known that the FRF was generated using very slow sine sweep rate from low to high frequency and the FFT operator was used to generate the FRF. The modified-Dobson method was applied to the measured FRF (red) and the modal parameters were polynomial-fitted and used for the equation (1), which is valid for well-separated mode as it was assumed for this blind case. The NFRS was generated and a synthesized nonlinear FRF was extracted and compared to the experimental one. Figure 3.7 shows an overlay between the experimental (red) and synthesized (black) FRFs. Given the excellent overlay, any engineer would be interested in the response for a lower or higher amplitude of vibrations. Hence, by changing the force amplitude, two FRF can be evaluated for low and high vibration amplitudes, both in blue colour. Unfortunately, there is no possible comparison for the two FRFs in blue as experimental data were unavailable. The reference paper [11] shows an academic example where such a comparison was possible and successful. Therefore, it is assumed that this regeneration approach for low and high amplitudes can still be valid.

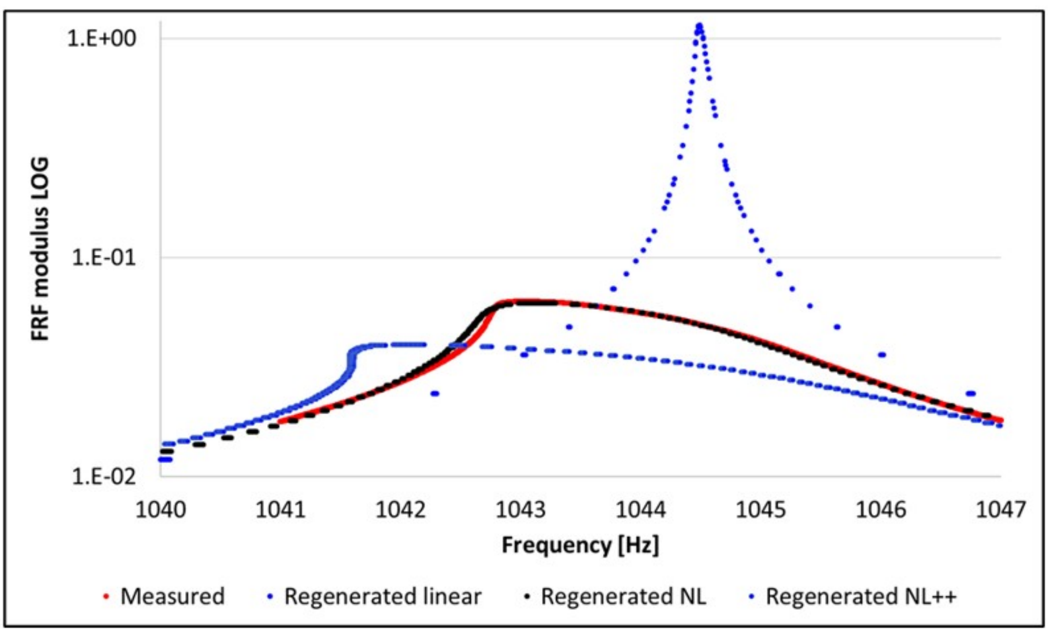

Fig. 6 Nonlinear FRFs measured (red) and regenerated in black and blue

Final remarks

This short paper summarizes the discovery reported in [11] by highlighting the fundamental steps for a test engineer to (i) measure, (ii) characterize, (iii) synthesize and (iv) compare the frequency nonlinear responses. These four steps are possible because a nonlinear frequency response curve can be built by using equation (1) as soon as the modal parameters of equation (1) are identified. Therefore, it is possible to evaluate for any desired level of excitation force a nonlinear frequency response which can be overlaid and compared to the experimental one. This final step is the most critical and valuable because the test engineer can evaluate if the identification process achieved the correct parameters. This is done is a matter of seconds by exploiting the fundamental equations of the linear modal analysis without using expensive and complex computational calculations.

Polynomials fit the modal amplitude-dependent parameters, and this process can intrinsically add some bias. Observing the response for much higher amplitudes than the maximum one measured is possible, but the risk is in the polynomial curves which might generate un-physical responses. Therefore, paying attention to simulation generated beyond the measured amplitudes of the responses is important.

A final remark is about the generation of the surface by using a force-displacement curve. Theoretically, any finite element method can be used to generate a fully reverse cycle of load by superimposing any desired deflection shape (at any resonance). It is a static analysis that allows the evaluation of a hysteresis curve. That curve can be used two ways, 1- to extract the static receptance and 2- to extract the damping from the hysteresis. Once these two parameters are extracted, an NFRS can be generated. It is straightforward to understand that any FE analyst can simulate a nonlinear frequency response given a stress model. Therefore, one can infer from the dynamic response (either linear or nonlinear) if the stress model needs better refinements, for instance, by increasing the damping by changing the material properties. Hence, the NFRS can become (i) a great design tool for analysts and (ii) a great experimental modal analysis tool for test engineers.

Conclusions

In conclusion, this paper provides some basic insights to the NFRS. It was shown that this surface can be generated by theoretical force-displacement full reverse cycle and by experimental identification of the nonlinear FRF. Both routes will meet simultaneously if a thorough experimental model validation is carried out. This paper and the primary reference [11] could not demonstrate it. Therefore, future work will be done to generate the NFRS from a theoretical route and validate the same surface by generating it using experimental measurements.

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