Chapter 14

Sigma-point Bootstrapping for Uncertainty Quantification in Subspace Identification for Non-Gaussian Estimates

Szymon Greś, Charikleia Stoura, Vasilis Dertimanis, and Eleni Chatzi

Abstract The estimation of modal properties of dynamic systems is a fundamental task in engineering. Since the estimates are inherently afflicted with uncertainties due to the unknown ambient excitation, measurement noise, and limited data length, quantification of the related estimation quality is paramount for applications to, e.g., Operational Modal Analysis and Structural Health Monitoring. In recent years, uncertainty quantification methods for subspace-based modal parameter estimates have been proposed based on the first-order sensitivity-based covariance propagation. Although proven successful in multiple applications, the first-order framework relies on the linearization of the modal estimates and is inaccurate when the estimates are non-Gaussian. In this paper, we propose a new method for covariance propagation that does not rely on Taylor series expansion and provides accurate probability distribution approximation for non-Gaussian modal estimates, e.g., when obtained from short data sets. The proposed approach is based on an unscented transformation of data sample covariance, which is efficiently used in a bootstrap-like approach for subspace identification. The performance of the method is illustrated on a simple mechanical system.

Keywords Unscented transform · uncertainty propagation · subspace identification · non-Gaussian estimates · short data lengths

Introduction

Covariance propagation and uncertainty quantification are fundamental tasks in modern engineering sciences. In recent years, uncertainty quantification methods for subspace identification methods have been proposed [1–4] based on the statistical delta method [5]. In this framework, the covariance of the estimated parameters is directly linked to the sample covariance of the vectorized data correlations through sensitivities obtained with the Taylor expansion of the considered identification algorithm. The first-order Taylor expansion provides an approximate method to propagate the covariance through a nonlinear function. Although the linearization is easy to implement, its precision is of the first-order, which by design does not encapsulate the high-order statistical information, whose significance is important for non-Gaussian estimates resulting from subspace identification problems with short data lengths or when estimated variables are close to the border of their distribution functions. While the second-order delta method can be used to approximate the asymptotic properties of the estimates at the borders [6, 7], it requires the knowledge of their true values, which limits its applications.

The unscented transformation is often used to approximate the mean and covariance of stochastic variables in nonlinear Bayesian filters. The key idea behind it is that with a fixed number of samples, the so-called sigma points, the estimate of a posterior mean and covariance of a variable is more accurate than using an approximation based on linearization or an arbitrary nonlinear transformation. Initially developed for Gaussian variables [8], which are fully characterized by their first two moments, it was found that the accuracy of the unscented transformation is reduced if the Gaussian assumption is not satisfied and the transformed function is highly nonlinear [9]. This sparked the development

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of unscented transforms that account for higher-order moments, like the skewness and kurtosis [10, 11]. While the unscented transformation is mostly applied in the Bayesian filtering context [12–14], its application relates to uncertainty quantification in general, where it can be applied to, e.g., uncertainty propagation in parameters of thermodynamic models [15].

In this work, we explore an application of the classic unscented transformation in the context of subspace identification to propagate the covariance of the sample covariance onto the estimates of modal parameters of a linear time-invariant (LTI) dynamic system. To this end, a set sigma points is constructed, the focus of which is to evaluate the considered random variable at the tail of its distribution. The chosen subspace identification algorithm is then applied to the set, and the unscented estimate is obtained by computing the mean of the transformed sigma points. Subsequently, its statistics are obtained using a kernel density estimate.

The paper is organized as follows: firstly, some background on the unscented transformation and subspace identification is given. Secondly, the sigma point uncertainty propagation framework is developed. Subsequently, the proposed method is validated on a simple numerical system.

Background

The purpose of this section is to first recap the main steps of a classic unscented transformation of a random variable and then recall the first steps of a generic subspace identification algorithm; for a thorough summary, the interested reader is referred to [9] and [16] for the unscented transform and subspace identification respectively.

Unscented Transformation

Let $\hat{x} = g(\hat{z}) \in \mathbb{R}^s$ denote an outcome of a nonlinear transformation $g(\cdot)$ of a random vector $\hat{z} \in \mathbb{R}^s$ with $z = \mathrm{E}[\hat{z}]$ and $\Sigma_z = \mathrm{E}[(\hat{z} - z)(\hat{z} - z)^T]$ respectively denoting the mean and the covariance of $\hat{z}$. The sample mean and sample covariance of $\hat{x}$ can be obtained with an unscented transform; to this end, $2s$ sigma points are defined

$$\begin{aligned} \chi_i &= z + \bar{z}_i, \quad i = 1 \dots 2s \\ \bar{z}_i &= \sqrt{s}\left(\Sigma_z^{1/2}\right)_i, \quad i = 1 \dots s \\ \bar{z}_{i+s} &= -\sqrt{s}\left(\Sigma_z^{1/2}\right)_i, \quad i = 1 \dots s, \end{aligned}$$

where χ_i denotes the i-th sigma point, the operator $(\cdot)_i$ denotes an i-th column of a matrix and $(\cdot)^{1/2}$ denotes a matrix square root. The sigma points are subsequently transformed

$$\mathcal{X}_i = g(\chi_i) \tag{1}$$

and the sample mean $\bar{x}$ and sample covariance Σ_x of the transformed sigma points writes

$$\bar{x} = \frac{1}{2s}\sum_{i=1}^{2s}\mathcal{X}_i, \quad \Sigma_x = \frac{1}{2s}\sum_{i=1}^{2s}\left(\mathcal{X}_i - \bar{x}\right)\left(\mathcal{X}_i - \bar{x}\right)^{\mathrm{T}}.$$

Both $\bar{x}$ and Σ_x accurately approximate the first two moments of x when z is asymptotically Gaussian [9]. For non-Gaussian z, approximations are accurate to at least the second-order [8].

Subspace Identification

Let $\{y_k\}_{k=1,\dots,N+p+q}$, $y_k \in \mathbb{R}^r$, denote samples of a response of an LTI mechanical system collected by r sensors. We assume that this response is generated by an independent and identically distributed zero-mean input process of finite fourth moments, which is persistently exciting the system modes. The dynamic system is also assumed to be stable, observable, and with distinct eigenvalues. The input measurements are assumed not available; however, the main contribution of this work can easily be generalized to input-output subspace methods.

Definition 1 (Data matrix) *Let $a_k \in \mathbb{R}^b$ be a discrete signal at time step k. The parameter p defines the 'past' and 'future' data horizons. For $0 \leq i \leq j \leq 2p-1$ the data matrix $\mathcal{A}_{i|j}$ writes*

$$\mathcal{A}_{i|j} = \frac{1}{\sqrt{N}} \begin{bmatrix} a_i & a_{i+1} & \dots & a_{i+N-1} \\ a_{i+1} & a_{i+2} & \dots & a_{i+N} \\ \vdots & \vdots & \vdots & \vdots \\ a_j & a_{j+1} & \dots & a_{j+N-1} \end{bmatrix} \in \mathbb{R}^{(j-i+1)b \times N}. \tag{2}$$

The first step in subspace identification is the estimation of the data projection matrix $\mathcal{H}$. To this end, we define data matrices $\mathcal{Y}^- = \mathcal{Y}_{0|p-1} \in \mathbb{R}^{ps \times N}$, $\mathcal{Y}^+ = \mathcal{Y}_{p|2p-1} \in \mathbb{R}^{ps \times N}$, where p denotes the time lag. The shape of $\hat{\mathcal{H}}$ depends on the chosen projection to nullify the influence of inputs and the instrumental variable to eliminate the noise. Hereafter, as an example, we use the output-only covariance-driven subspace identification algorithm with the projection matrix

$$\hat{\mathcal{H}} = \mathcal{Y}^+ (\mathcal{Y}^-)^T . \tag{3}$$

For a chosen model order, the estimates of modal parameters are obtained as a function of $\hat{\mathcal{H}}$

$$\hat{\eta} = g(\hat{\mathcal{H}}), \text{ with } \hat{\eta} = \begin{bmatrix} \hat{f} \\ \hat{\zeta} \\ \text{vec}(\hat{\varphi}) \end{bmatrix}$$

where $\hat{f}$, $\hat{\zeta}$, and $\hat{\varphi}$ are the estimated natural frequencies, damping ratios, and mode shapes, respectively. $g(\cdot)$ denotes the chosen subspace identification algorithm.

Sigma point bootstrap for uncertainty propagation

In this section, an application of the unscented transformation for uncertainty propagation in subspace identification is outlined. Let $\hat{\Sigma}_{\mathcal{H}}$ be a consistent estimate of the sample covariance of the vectorized $\hat{\mathcal{H}}$ obtained from n_b blocks of available data

$$\hat{\Sigma}_{\mathcal{H}} = \frac{1}{n_b(n_b-1)} \sum_{j=1}^{n_b} \text{vec}\left(\hat{\mathcal{H}}_j - \hat{\mathcal{H}}\right) \text{vec}\left(\hat{\mathcal{H}}_j - \hat{\mathcal{H}}\right)^{\mathrm{T}}, \tag{4}$$

where $\hat{\mathcal{H}}_j$ denotes a Hankel matrix computed on $j = 1 \dots n_b$ data blocks. The sigma points are defined as

$$\begin{aligned} \hat{\mathcal{H}}_i &= \hat{\mathcal{H}} + \bar{\mathcal{H}}_i, \quad i = 1 \dots 2n_b \\ \tilde{\mathcal{H}}_i &= \sqrt{k_i} \left(\Sigma_z^{1/2}\right)_i, \quad i = 1 \dots n_b \\ \tilde{\mathcal{H}}_{i+n_b} &= -\sqrt{k_i} \left(\Sigma_z^{1/2}\right)_i, \quad i = 1 \dots n_b, \end{aligned}$$

where k_i denotes an i-th element of $k = \begin{bmatrix} 1 & 3 & \dots & 2n_b \end{bmatrix}$ and $\bar{\mathcal{H}}_i = \text{vec}^{-1}(\tilde{\mathcal{H}}_i)$ where $\text{vec}^{-1}(\cdot)$ denotes an inverse vectorization operator. Note that the weighting strategy employed herein aims at evaluating the sigma points at the tail of the sample distribution of $\hat{\mathcal{H}}$, which in principle is different than in the classic unscented transformation. Once the samples of the modal parameters are obtained from the transformed sigma points

$$\hat{\eta}_i = g(\hat{\mathcal{H}}_i),$$

the unscented estimate can be computed

$$\hat{\eta}^{\text{un}} = \frac{1}{2n_b} \sum_{i=1}^{2n_b} \hat{\eta}_i$$

and the joint probability distribution of the unscented estimates can be established through the kernel density estimation

$$\hat{f}(\eta) = \frac{1}{n_p h} \sum_{i}^{n_p} K\left(\frac{\eta - \bar{\eta}}{h}\right),$$

where $\bar{\eta}$ are equally-spaced points that cover the range of $\hat{\eta}_i$ for $i = 1 \dots 2n_b$, n_p is the number of the points, K is the kernel smoothing function, and h is the bandwidth. In this work, a Gaussian kernel with 100 data points is used.

Numerical Validation

The uncertainty propagation strategy outlined in the previous section is validated on a 6 DOF mechanical chain-like system that for any consistent set of units is modeled with spring stiffness $k_1 = k_3 = k_5 = 100$ and $k_2 = k_4 = k_6 = 200$, mass $m_i = 1/20$, and a non-proportional damping matrix

$$C = \begin{bmatrix} 0.2007 & -0.0236 & 0.0595 & 0.1170 & 0.0736 & 0.1341 \\ -0.0236 & 0.0192 & -0.1054 & -0.0748 & -0.1617 & -0.0773 \\ 0.0595 & -0.1054 & 0.1263 & -0.0297 & -0.0843 & 0.0286 \\ 0.1170 & -0.0747 & -0.0297 & 0.2570 & -0.0748 & 0.1227 \\ 0.0735 & -0.1617 & -0.0842 & -0.0748 & -0.0601 & -0.1141 \\ 0.1341 & -0.0773 & 0.0286 & 0.1226 & -0.1142 & 0.2558 \end{bmatrix} \cdot 10^{-4}.$$

The chain is illustrated in Figure 1.

The system is excited by a white noise signal persistently acting at all DOFs. The acceleration outputs are simulated at DOFs 1, 3, and 5 at a sampling frequency of 50 Hz for 200 seconds, generating 10.000 data points. An additional white measurement noise with 5% of the standard deviation of the output is added to each response measurement. The computations are performed in a Monte Carlo setup with 1000 realizations of the input-output data. The output-only covariance-driven subspace system identification is used, and the estimates are obtained at system order 12, using 15 time lags for the computation of the Hankel matrix.

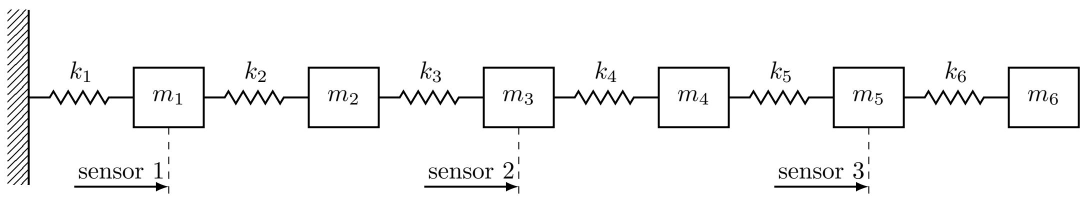

Fig. 1 6 DOF chain system sketch.

The sample covariance of the Hankel matrix is obtained by dividing the output data to 200 blocks, resulting in 400 sigma points for uncertainty propagation. The histograms of the classic subspace-based natural frequency and damping ratio estimates, and the corresponding estimates obtained from one realization of the sigma point sampling strategy are illustrated in Figures 2 and 3.

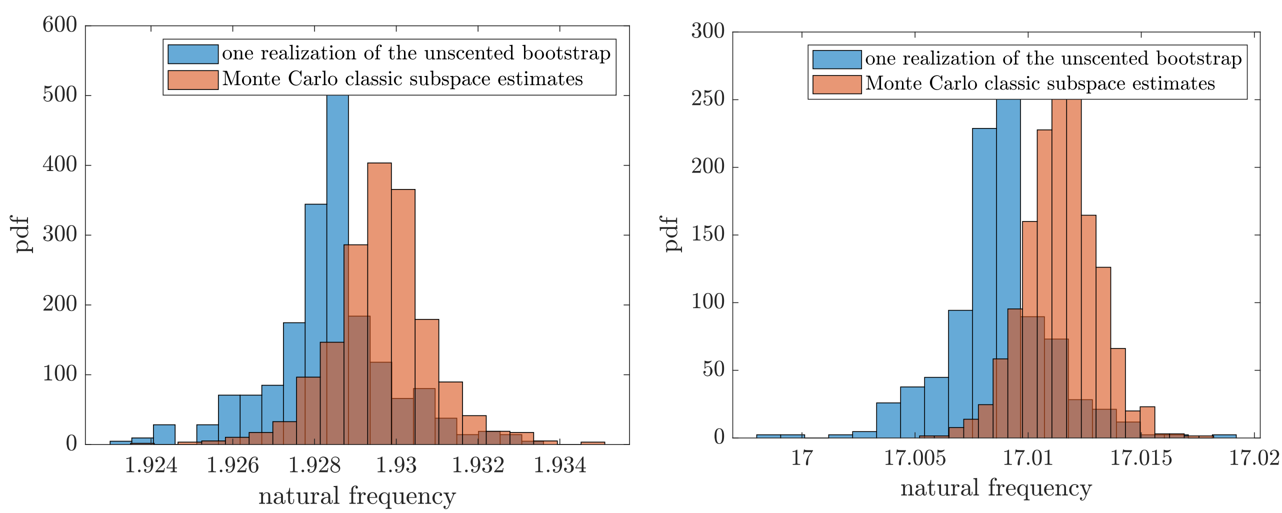

Fig. 2 Natural frequency estimates-first mode (left) and sixth mode (right).

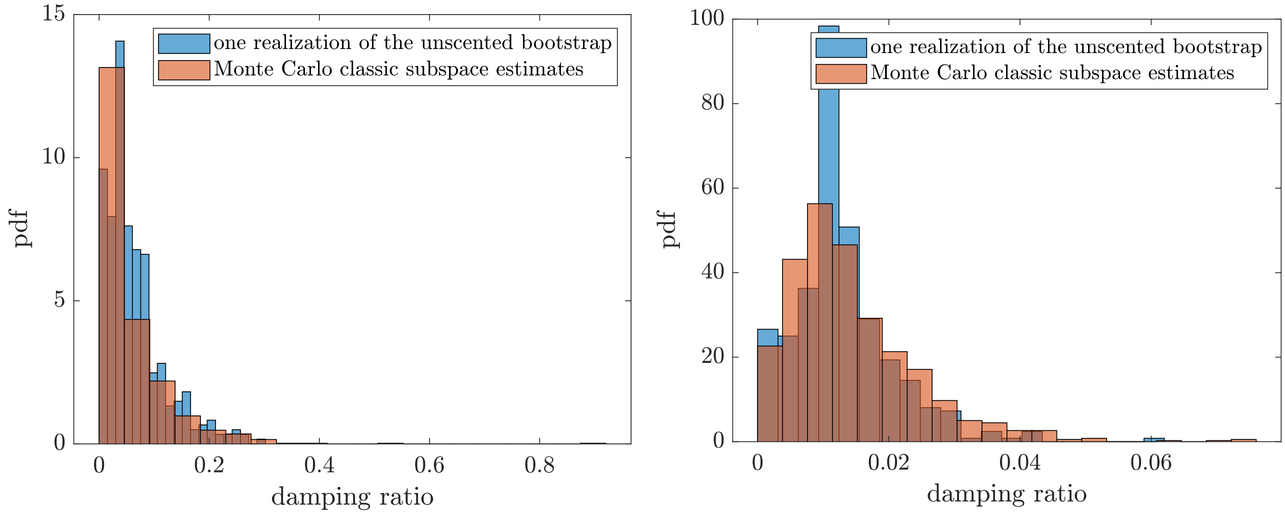

Fig. 3 Damping ratio estimates-first mode (left) and sixth mode (right).

It can be viewed that the independent realizations of the natural frequency and damping ratio estimates yield histograms illustrating their distributions; while the shape of the natural frequency histograms for both approaches resembles a Gaussian probability density function, the damping ratio estimates accumulate at the border of the distribution at 0, where the Gaussian assumption is clearly violated. While it is expected that the classic first-order perturbation approach is suitable for uncertainty quantification of a Gaussian distributed variable, its application for uncertainty assessment of estimates that are non-Gaussian is not adequate.

Figure 2 illustrates that although the mean of the histogram corresponding to the classical subspace-based natural frequency estimates does not coincide with the mean of the histogram corresponding to one realization of the unscented bootstrap, the spread of the histograms is similar. It is not expected that the means will match in this case; a match between the two should be expected when the mean of the unscented estimates is considered, which is illustrated in Figures 4 and 5.

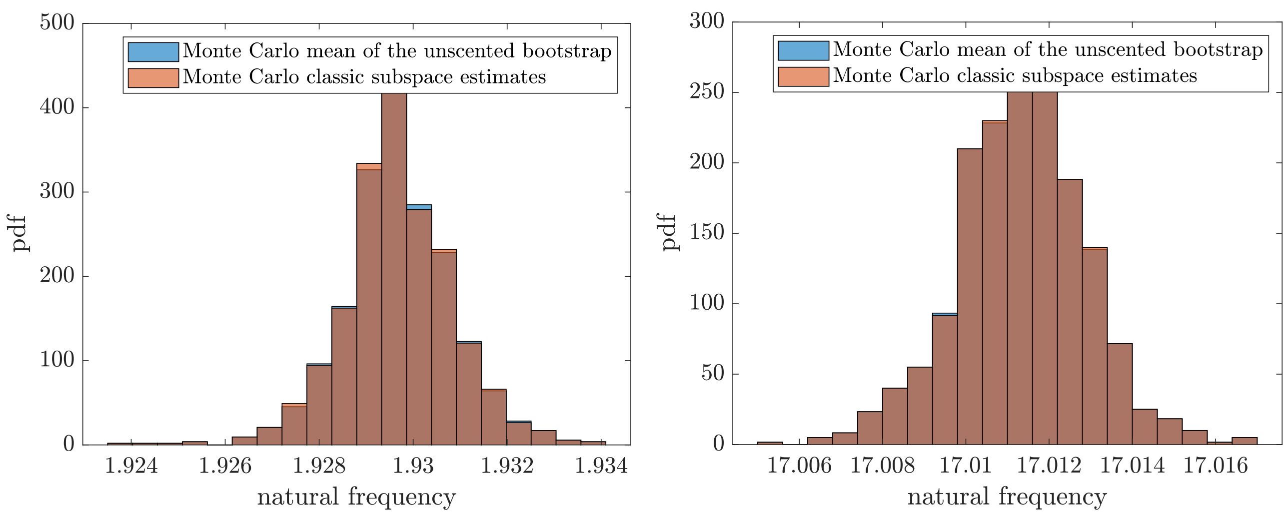

Fig. 4 Natural frequency estimates-first mode (left) and sixth mode (right).

Figures 2 and 4 suggest that the natural frequency estimates are possibly asymptotically Gaussian distributed; therefore, hereafter, we will focus on comparing the standard deviations obtained using the classic first-order perturbation approach and the sample standard deviations obtained from the unscented bootstrap, omitting the kernel density estimation step. Histograms of both quantities are illustrated in Figure 6. It can be viewed that the mean of the standard deviation estimates obtained using the first-order perturbation approach and the sample standard deviations of the

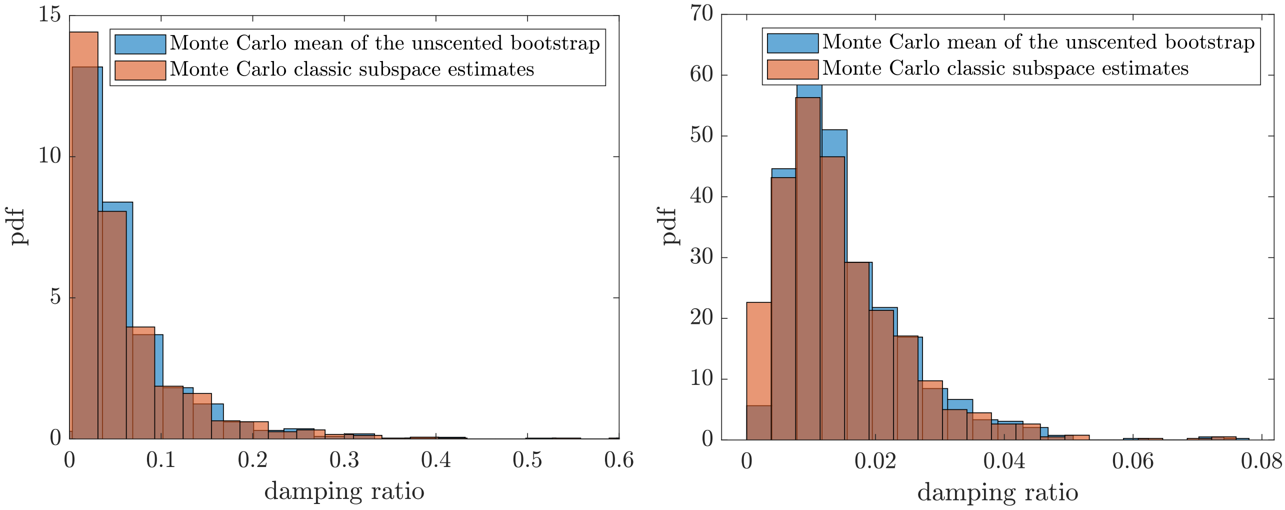

Fig. 5 Damping ratio estimates- the first mode (left), the sixth mode (right) .

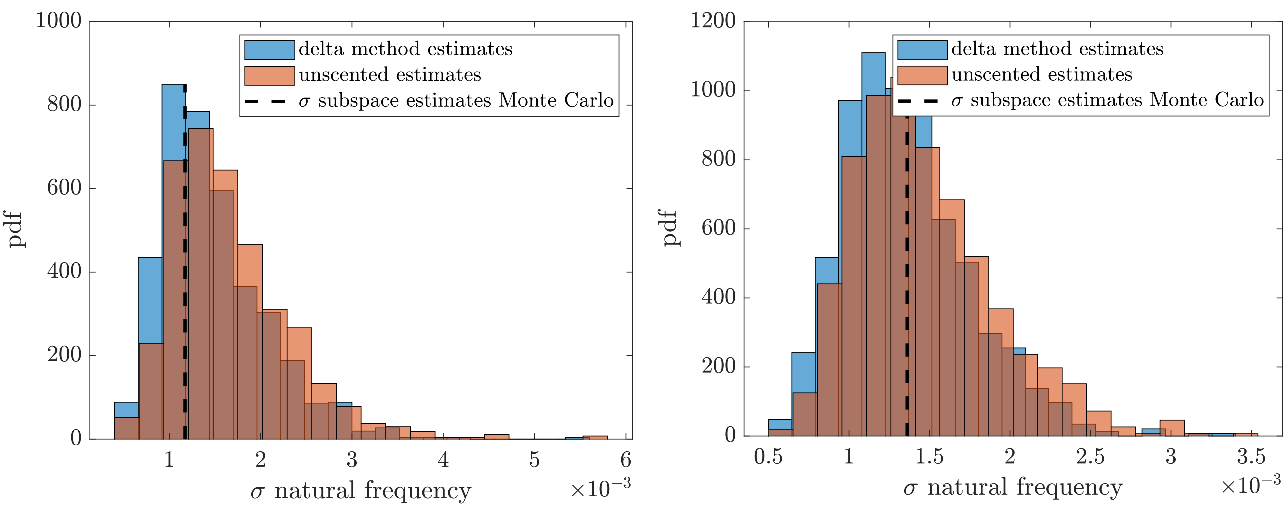

Fig. 6 Standard deviation (σ) of natural frequency estimates-first mode (left) and sixth mode (right).

unscented estimates match well with each other and coincide with the standard deviation of the classic subspace Monte Carlo histograms.

The motivation behind the proposed method is the estimation of the statistical properties of non-Gaussian subspace-based estimates. This is showcased in the estimates of damping ratios, which are clearly non-Gaussian due to the significant identification errors stemming from short data lengths. Therefore, to validate the proposed approach, the average of the probability density functions obtained with the first-order approach and the unscented bootstrap are compared to the histogram of the estimates in Figure 7. It can be viewed that the average probability density function obtained via kernel density estimate matches well with the histograms of the damping ratio estimates despite the estimates being non-Gaussian.

The proposed method clearly outperforms the first-order approach in uncertainty quantification of non-Gaussian variables; however, the price for the accuracy is the numerical efficiency of the bootstrap: using Matlab 2020b on AMD Ryzen 7 5800H CPU with 64GB of RAM, one evaluation of an efficient implementation of the first-order approach takes 0.01s and a parallelized unscented bootstrap takes 0.49s.

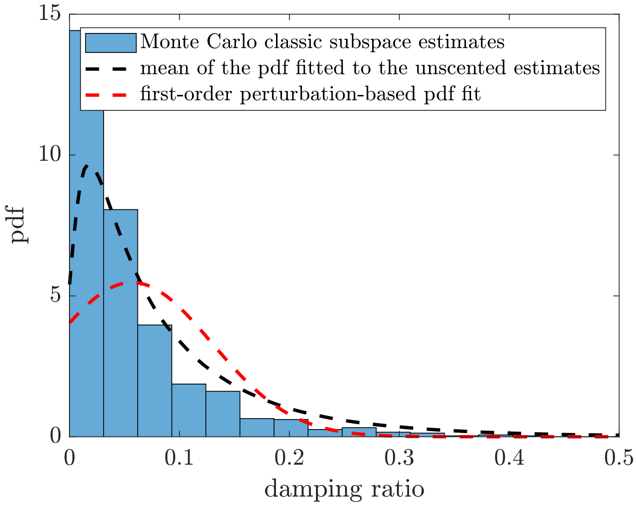

Fig. 7 Uncertainty of damping ratio estimates-first mode (left) and sixth mode (right).

Conclusions

In this work, we present an application of the unscented transformation for the estimation of the probability density functions of the modal parameter estimates obtained in subspace identification. The performance of the proposed method is equivalent to the classic first-order sensitivity-based approach for estimates following a Gaussian distribution. However, it is shown that classic methods fail to approximate the statistical properties of non-Gaussian parameters, whereas the proposed unscented transformation provides a more accurate approximation. The cost of this increased accuracy is a reduction in the computational efficiency of the bootstrap.

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