Chapter 13
Revisiting the Dual Admittance-Based Quasi-Static Formulation for the Identification of Linear Joint Dynamics with Dynamic Substructuring

Francesco Trainotti, M. Brons, S. Klaassen, D. J. Rixen

Abstract Accurately capturing joint characteristics in mechanical systems is essential for structural analysis, design optimization and condition monitoring. Substructuring decoupling offers a straightforward procedure to isolate the linear(-ized) joint dynamics in terms of frequency-based dynamic stiffness or admittance functions. Assuming a quasi-static representation of the joint, identified therein as no mass and no cross-coupling effects, a variety of formulations with (seemingly) different roots can be found in literature that differ in several aspects, such as the type, location and amount of impedances/transfer functions employed, as well as the (sub)-components required. This article presents a concise review of popular quasi-static identification formulations. Additionally, it provides an analysis of the discrepancies and equivalence between these formulations, by revisiting the dual admittance-based framework.

Keywords Joint identification · Experimental substructuring · Inverse substructuring · Quasi-static joint

Introduction

Mechanical joints are fundamental components in assembled mechanical systems, significantly impacting their dynamic performance, structural integrity, and noise and vibration characteristics. Accurate identification of joint properties is essential for effective structural analysis, optimized design, and reliable condition monitoring. With advancements in measurement technology and the adoption of Experimental Substructuring techniques [1, 2], there has been a growing interest in efficient procedures to isolate and identify joint dynamics.

A key consideration in joint modeling is the choice between dynamic and quasi-static representations. Dynamic joint modeling treats the joint as a full dynamic component. In contrast, quasi-static joint modeling simplifies the joint to a compliant linkage between interfaces, neglecting inertial effects. This assumption has gained popularity due to its simplicity and practicality, making it suitable for a wide range of engineering applications. One important advantage of quasi-static modeling is its applicability to measurements of the joint in its assembled configuration, often referred to as ’in-situ’ measurements.

Another important aspect is the formulation approach, which can be categorized into primal and dual methods. Primal formulations impose compatibility constraints by assuming that the interface degrees of freedom are identical and perfectly aligned. On the other hand, dual formulations allow discrepancies between the interface degrees of freedom and aim to minimize these gaps, providing greater flexibility, especially in systems where perfect alignment is difficult to achieve.

Focusing on quasi-static identification approaches, several studies have contributed to the development and refinement of these methods. Primal quasi-static joint identification techniques have been explored in works such as [3, 4, 5, 6, 7, 8]. Dual quasi-static methods have been extensively investigated in works like [9, 10, 11, 12, 13, 14].

Despite the variety of formulations available, discrepancies often arise due to differences in the types, locations, and quantities of impedances or transfer functions used, associated with interface or internal measurements of the required (sub)-components. Such variations make it challenging to select the most appropriate method for a given application.

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This paper presents a short review of popular quasi-static joint identification formulations, with a focus on both primal and dual approaches. By analyzing the discrepancies and commonalities between these methods, along with a novel revisitation of the dual admittance-based framework, the aim is to provide a unified perspective on these techniques.

The paper is structured as follows: section 2 provides the theoretical background on quasi-static identification, with a focus on two popular formulations within the primal and dual domains. Then, section 3 introduces insights into the equivalence of these methods, including a novel derivation of the joint isolation process in admittance form in the dual sense. Finally, section 4 offers concluding remarks.

Quasi-static Substructuring Identification

The goal of the substructuring characterization is to isolate the dynamic effects of a joint component J from the connecting components A and B defining the linear(-ized) time invariant assembly AJB. This is illustrated in fig. 1. A partitioning of the measurements between internal and interface is realized. For simplicity, we assume that the 'experimental' interface is defined such that there is a compatible set of degrees of freedom between the joint and the two components that ensures full controllability and observability of the interface and joint dynamics. Furthermore, we assume that the joint is not easily accessible in practice, so that the extraction of its dynamical properties should be done via indirect measurements of the assembled system.

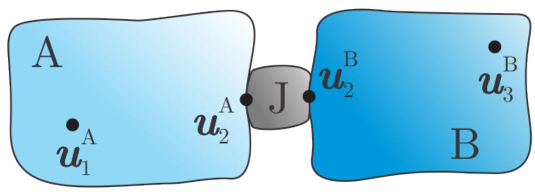

Fig. 1 Illustration of the assembled system connecting components A and B to a joint J. $\mathbf{u}$ refers to displacement, where the subscripts $\star_1$, $\star_3$ indicate internal degrees of freedom and $\star_2$ represents the interface.

The substructuring isolation can be classified on the basis of the following general features:

- Dynamic joint or quasi-static joint
- Primal approach or dual approach
- Using interface measurements or internal measurements

The focus of this article is on a quasi-static identification performed via a primal or a dual formulation with solely interface measurements. A quasi-static joint is herein defined as a mass-less compliant mechanism with no cross-coupling between the connected interfaces' degrees of freedom.

A quasi-static joint parametrized with a simple spring-damper mechanism is represented in fig. 2.

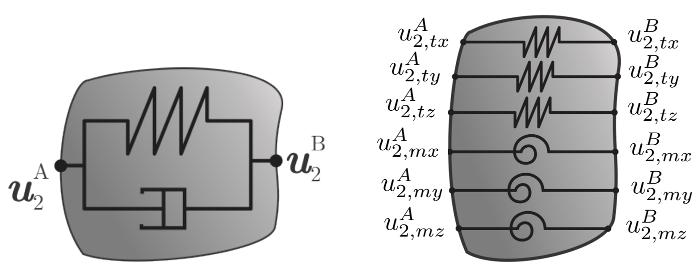

Fig. 2 Illustration of the quasi-static joint. Left: Representation of the joint as a massless component. Right: Example illustrating the DoF to DoF topology of the joint, i.e. the no cross-coupling assumption.

Let us define the admittances of the assembled system and sub-components at the defined collocated interface degrees of freedom as $\mathbf{Y}_{22}^{AJB}$ and $\mathbf{Y}_{22}^{A|B} = \begin{bmatrix} \mathbf{Y}_{22}^{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{Y}_{22}^{B} \end{bmatrix}$ respectively. Note that both matrices have the same dimension, given the quasi-static assumption. The corresponding impedances are obtained by simple inversion assuming the same number of output and input degrees of freedom as $\mathbf{Z}_{22}^{AJB} = \left(\mathbf{Y}_{22}^{AJB}\right)^{-1}$ and $\mathbf{Z}_{22}^{A|B} = \left(\mathbf{Y}_{22}^{A|B}\right)^{-1}$.

A global set of ordered interface degrees of freedom $\mathbf{u}_2$, external forces $\mathbf{f}_2$ and interface forces $\mathbf{g}_2$ are also introduced, stacking together the corresponding quantities from components A and B.

Quasi-static primal formulation with impedances: Inverse Substructuring

A primal formulation of substructuring, incorporating the equations of motion and interface conditions, and assuming exact compatibility, is expressed in impedance form as follows [1]:

$$\mathbf{Z}_{22}^{A|B}\mathbf{u}_2 = \mathbf{f}_2 + \mathbf{g}_2 \tag{1a}$$

$$\mathbf{u}_2 = \mathbf{L}\mathbf{q}_2 \tag{1b}$$

In this formulation, the compatibility condition is automatically satisfied by reducing the global interface coordinate, $\mathbf{u}_2$, to a minimal matching set of solving coordinates ,$\mathbf{q}_2$, through the localization Boolean operator $\mathbf{L}$.

Integrating eq. (1b) into eq. (1a) and ensuring equilibrium by left-projecting the equations of motion using $\mathbf{L}$ leads to the assembled configuration.

A primal formulation naturally favors the use of impedances, as it exposes the topological connections between components. By incorporating a dynamic joint component $\begin{bmatrix} \mathbf{Z}^J_{2^A2^A} & \mathbf{Z}^J_{2^A2^B} \\ \mathbf{Z}^J_{2^B2^A} & \mathbf{Z}^J_{2^B2^B} \end{bmatrix}$ into the global uncoupled impedance of eq. (1a), the dynamic stiffness of the assembled system can be expressed as:

$$\mathbf{Z}_{22}^{AJB} = \begin{bmatrix} \mathbf{Z}^A_{22} + \mathbf{Z}^J_{2^A2^A} & \mathbf{Z}^J_{2^A2^B} \\ \mathbf{Z}^J_{2^B2^A} & \mathbf{Z}^B_{22} + \mathbf{Z}^J_{2^B2^B} \end{bmatrix} \tag{2}$$

The joint dynamics are directly accessible from the off-diagonal terms of eq. (2). In an experimental setting, the isolation of the joint properties is thus implicitly achieved by inverting the assembled admittance model $\mathbf{Y}_{22}^{AJB}$. This allows for direct access to the joint dynamics within the system. This identification procedure is referred to here as Inverse Substructuring.

Notably, no assumptions regarding the joint properties have been made so far. However, by explicitly imposing the characteristics discussed in the introduction of section 2, namely the absence of mass and cross-coupling, the following quasi-static joint model can be reconstructed:

$$\left(\mathbf{Z}^J_{2^A2^A} \approx -\mathbf{Z}^J_{2^A2^B} \approx -\mathbf{Z}^J_{2^B2^A} \approx \mathbf{Z}^J_{2^B2^B}\right) = \mathbf{Z}^J_c, \qquad \mathbf{Z}^J = \begin{bmatrix} \mathbf{Z}^J_c & -\mathbf{Z}^J_c \\ -\mathbf{Z}^J_c & \mathbf{Z}^J_c \end{bmatrix} \tag{3}$$

The approximation symbol $\approx$ denotes the application of symmetry in the context of experimental measurements, while the property itself arises from the topological construction at the theoretical level.

Note that the literature distinguishes between isolating the joint dynamics via eq. (2) and fully determining a joint model as in eq. (3). Due to the topological construction, the dynamics of the joint, as well as those of the sub-components, are isolated and identified solely based on the dynamics of the assembled system.

Quasi-static dual formulation with admittances: LM-FBS with weakened interface

A dual formulation of the substructuring operation is expressed in admittance form as follows [1]:

$$\mathbf{u}_2 = \mathbf{Y}_{22}^{A|B}(\mathbf{f}_2 - \mathbf{B}^T\boldsymbol{\lambda}) \tag{4a}$$

$$\mathbf{B}\mathbf{u}_2 = \mathbf{0} \tag{4b}$$

In this formulation, the compatibility condition is strictly enforced (no interface flexibility) using a Boolean operator $\mathbf{B}$, while the equilibrium condition is inherently satisfied by introducing Lagrange multipliers $\boldsymbol{\lambda}$. These multipliers represent the intensity of the interface forces $\mathbf{g}_2$ and serve as solving variables.

A massless joint is introduced as a relaxation of the connection between the components. Mathematically, this is expressed as:

$$\mathbf{B}\boldsymbol{u}_2 = \Delta\boldsymbol{u}_2^J = \mathbf{Y}_c^J\boldsymbol{\lambda} \tag{5}$$

A gap $\Delta\boldsymbol{u}$ is now relaxing the compatibility condition via the quasi-static action of the joint dynamics $\mathbf{Y}_c^J$. Note that quasi-staticity at this point is solely referred to the absence of a mass term disrupting the equilibrium between the two sides of the interface.

By solving eq. (4a) using the relaxed compatibility in eq. (5), the expression for the assembled system admittance is retrieved:

$$\mathbf{Y}_{22}^{AJB} = \left[\mathbf{I} - \mathbf{Y}_{22}^{A|B}\mathbf{B}^T\left(\mathbf{B}\mathbf{Y}_{22}^{A|B}\mathbf{B}^T + \mathbf{Y}_c^J\right)^{-1}\mathbf{B}\right]\mathbf{Y}_{22}^{A|B} \tag{6}$$

which is referred as Lagrange Multiplier-Frequency Based Substructuring (LM-FBS) with weakened interface [14].

The joint admittance can be isolated by rearranging eq. (6):

$$\mathbf{Y}_c^J = \mathbf{B}\mathbf{Y}_{22}^{A|B}\mathbf{B}^T(\mathbf{B}(\mathbf{Y}_{22}^{AJB} - \mathbf{Y}_{22}^{A|B})\mathbf{B}^T)^{-1}\mathbf{B}\mathbf{Y}_{22}^{A|B}\mathbf{B}^T - \mathbf{B}\mathbf{Y}_{22}^{A|B}\mathbf{B}^T \tag{7}$$

To maintain consistency with the definition of the quasi-static joint in section 2, the final step involves selecting a specific topology for the joint by defining the compatibility Boolean matrix as $\mathbf{B} = \begin{bmatrix}-\mathbf{I} & \mathbf{I}\end{bmatrix}$. This choice enforces a DoF to DoF connection between the two sides of the interface, effectively eliminating any cross-coupling term. As a result, this assumption allows to write the following explicit expression:

$$\mathbf{Y}_c^J = \left(\mathbf{Y}_{22}^A + \mathbf{Y}_{22}^B\right)\left(\mathbf{Y}_{2^A2^A}^{AJB} + \mathbf{Y}_{2^B2^B}^{AJB} - \mathbf{Y}_{2^A2^B}^{AJB} - \mathbf{Y}_{2^B2^A}^{AJB} - \mathbf{Y}_{22}^A - \mathbf{Y}_{22}^B\right)^{-1}\left(\mathbf{Y}_{22}^A + \mathbf{Y}_{22}^B\right) - \left(\mathbf{Y}_{22}^A + \mathbf{Y}_{22}^B\right) \tag{8}$$

An astute reader will notice the following:

- To match the topological assumptions implicitly made in the primal joint derivation, a specific choice of the compatibility function $\mathbf{B} = \begin{bmatrix}-\mathbf{I} & \mathbf{I}\end{bmatrix}$ is required. While more complex connections with cross-coupling spring-like terms are possible, non-Boolean shape functions for $\mathbf{B}$ can also be used to model more advanced mechanisms, such as directional constraints. Note that one can also devise the corresponding topology for the isolation of the joint component in primal form, offering an alternative to the quasi-static joint in eq. (3).
- Although both the primal impedance and dual admittance formulations integrate equivalent joint topologies, the result in eq. (8) appears to necessitate knowledge of the dynamics of the isolated components. This requirement obscures the advantageous features of the intended quasi-static principle, which favors a practical and efficient identification method, specifically, the measurement of the joint in its assembled configuration (so called 'in-situ'). As a consequence, the mathematical equivalence between the primal and dual derivations is not readily evident.

The next section will address the open questions raised in the second bullet point.

On the equivalence between primal and dual quasi-static formulations

To demonstrate the equivalence of the quasi-static formulations in section 2.1 and section 2.2, the most straightforward approach is to begin with the dual quasi-static equations in eq. (4a) and eq. (5), and invert all the admittance terms. This leads to the following system:

$$\mathbf{Z}_{22}^{A|B}\mathbf{u}_2 = \mathbf{f}_2 - \mathbf{B}^T\boldsymbol{\lambda} \tag{9a}$$

$$\mathbf{Z}_c^J\mathbf{B}\boldsymbol{u}_2 = \mathbf{Z}_c^J\Delta\boldsymbol{u}_2^J = \boldsymbol{\lambda} \tag{9b}$$

By solving eq. (9) for $\boldsymbol{\lambda}$, an impedance-based formulation for the interface force-displacement relationship is derived:

$$\left[\mathbf{Z}_{22}^{A|B} + \mathbf{B}^T\mathbf{Z}_c^J\mathbf{B}\right]\mathbf{u}_2 = \mathbf{f}_2 \tag{10}$$

Analogous to section 2.2, the application of $\mathbf{B} = \begin{bmatrix}-\mathbf{I} & \mathbf{I}\end{bmatrix}$ leads to:

$$\left[\begin{bmatrix}\mathbf{Z}_{22}^A & \mathbf{0}\\ \mathbf{0} & \mathbf{Z}_{22}^B\end{bmatrix} + \begin{bmatrix}\mathbf{Z}_c^J & -\mathbf{Z}_c^J\\ -\mathbf{Z}_c^J & \mathbf{Z}_c^J\end{bmatrix}\right]\mathbf{u}_2 = \mathbf{f}_2 \tag{11}$$

This leads to the isolation of the joint and sub-components impedances, demonstrating the equivalence of the dual derivation with the primal (see eq. (2) and eq. (3)).

What remains unanswered is the formulation of the joint (and sub-components) admittances solely as a function of the assembled system admittance, as this aspect remains somewhat unclear in eq. (8). To the best of the author's knowledge, while a primal-based derivation exists in the literature, a dual-based derivation has not yet been published in full form. A concise summary of the primal derivation by [4] is provided in section 3.1, followed by the presentation of a dual-based expression in section 3.2.

Admittance-based primal derivation

A concise summary of the derivation steps from [4] is as follows:

1. The primal quasi-static impedance-based formulation as in eq. (2) and eq. (3) is used as the starting point.
2. The block matrix inversion formula in eq. (34) is applied to obtain the assembled system dynamics in admittance notation $\mathbf{Y}_{22}^{AJB} = (\mathbf{Z}_{22}^{AJB})^{-1}$
3. A system of four equations (one per block element of the assembled system $\mathbf{Y}_{22}^{AJB}$) is solved to find the three unknowns $\mathbf{Y}_{22}^{A}$,$\mathbf{Y}_{22}^{B}$,$\mathbf{Y}_{c}^{J}$ as a function of the assembled admittances $\mathbf{Y}_{2_A2_A}^{AJB}$, $\mathbf{Y}_{2_A2_B}^{AJB}$, $\mathbf{Y}_{2_B2_A}^{AJB}$, $\mathbf{Y}_{2_B2_B}^{AJB}$.

After lengthly derivations (applying matrix inversion properties as described in appendix A, along with the system's symmetry properties and various equation manipulations), this results in the following solutions for the isolated subcomponents and the joint:

$$\mathbf{Y}_{22}^{A} = \mathbf{Y}_{2^A2^B}^{AJB}\left(\mathbf{Y}_{2^B2^B}^{AJB} - \mathbf{Y}_{2^A2^B}^{AJB}\right)^{-1}\left(\mathbf{Y}_{2^A2^A}^{AJB}(\mathbf{Y}_{2^B2^A}^{AJB})^{-1}\mathbf{Y}_{2^B2^B}^{AJB} - \mathbf{Y}_{2^A2^B}^{AJB}\right) \tag{12a}$$

$$\mathbf{Y}_{22}^{B} = \left(\mathbf{Y}_{2^A2^A}^{AJB}\left(\mathbf{Y}_{2^B2^A}^{AJB}\right)^{-1}\mathbf{Y}_{2^B2^B}^{AJB} - \mathbf{Y}_{2^A2^B}^{AJB}\right)\left(\mathbf{Y}_{2^A2^A}^{AJB} - \mathbf{Y}_{2^A2^B}^{AJB}\right)^{-1}\mathbf{Y}_{2^A2^B}^{AJB} \tag{12b}$$

$$\mathbf{Y}_{c}^{J} = (\mathbf{Z}_{c}^{J})^{-1} = \mathbf{Y}_{2^A2^A}^{AJB}\left(\mathbf{Y}_{2^B2^A}^{AJB}\right)^{-1}\mathbf{Y}_{2^B2^B}^{AJB} - \mathbf{Y}_{2^A2^B}^{AJB} \tag{12c}$$

It is evident that there is an explicit and sole dependence on the dynamics of the assembled system. This result will be taken as reference for the dual-based derivation in section 3.2.

Further, upon inspecting the result in eq. (12c), the following observation can be made:

$$\mathbf{Y}_{c}^{J} = (\mathbf{Z}_{c}^{J})^{-1} \approx (-\mathbf{Z}_{2^A2^B}^{AJB})^{-1} = -\mathbf{Y}_{2^A2^B}^{AJB,\text{cond}} = \mathbf{Y}_{2^A2^A}^{AJB}\left(\mathbf{Y}_{2^B2^A}^{AJB}\right)^{-1}\mathbf{Y}_{2^B2^B}^{AJB} - \mathbf{Y}_{2^A2^B}^{AJB} \tag{13}$$

The admittance form of the joint dynamics can thus be derived using the condensation rule (indicated by the superscript $\star^{\text{cond}}$), leveraging the topology of the primal formulation in eq. (2).

Admittance-based dual derivation

Starting from the quasi-static dual formulation in eq. (6), assuming the topology given by $\mathbf{B} = \begin{bmatrix}-\mathbf{I} & \mathbf{I}\end{bmatrix}$:

$$\mathbf{Y}_{22}^{AJB} = \begin{bmatrix}\mathbf{Y}_{22}^{A} & \mathbf{0}\\ \mathbf{0} & \mathbf{Y}_{22}^{B}\end{bmatrix} - \begin{bmatrix}-\mathbf{Y}_{22}^{A}\\ \mathbf{Y}_{22}^{B}\end{bmatrix}\left(\mathbf{Y}_{22}^{A} + \mathbf{Y}_{22}^{B} + \mathbf{Y}_{c}^{J}\right)^{-1}\begin{bmatrix}-\mathbf{Y}_{22}^{A} & \mathbf{Y}_{22}^{B}\end{bmatrix} \tag{14}$$

From here on, the notation will be simplified using $\mathbf{Y}^{\text{tot}} = \mathbf{Y}_{22}^{A} + \mathbf{Y}_{22}^{B} + \mathbf{Y}_{c}^{J}$ and omitting the subscript $\star_{22}$ for $\mathbf{Y}_{22}^{A}$ and $\mathbf{Y}_{22}^{B}$.

Similarly to the primal-based steps described in section 3.1, a system of four equations can be derived from the block elements of eq. (14):

$$\mathbf{Y}_{2^A2^A}^{AJB} = \mathbf{Y}^{A} - \mathbf{Y}^{A}\left(\mathbf{Y}^{\text{tot}}\right)^{-1}\mathbf{Y}^{A} \tag{15a}$$

$$\mathbf{Y}_{2^A2^B}^{AJB} = \mathbf{Y}^{A}\left(\mathbf{Y}^{\text{tot}}\right)^{-1}\mathbf{Y}^{B} \tag{15b}$$

$$\mathbf{Y}_{2^B2^B}^{AJB} = \mathbf{Y}^{B} - \mathbf{Y}^{B}\left(\mathbf{Y}^{\text{tot}}\right)^{-1}\mathbf{Y}^{B} \tag{15c}$$

$$\mathbf{Y}_{2^B2^A}^{AJB} = \mathbf{Y}^{B}\left(\mathbf{Y}^{\text{tot}}\right)^{-1}\mathbf{Y}^{A} \tag{15d}$$

Note that, thanks to the symmetry of the admittances, it is true that $\mathbf{Y}_{2^A2^B}^{AJB} = \left(\mathbf{Y}_{2^B2^A}^{AJB}\right)^{T}$.

The stated goal is to find the three unknowns $\mathbf{Y}^{A}$, $\mathbf{Y}^{B}$, $\mathbf{Y}_{c}^{J}$ as a function of the admittances of the assembled system $\mathbf{Y}_{2_A2_A}^{AJB}$, $\mathbf{Y}_{2_A2_B}^{AJB}$, $\mathbf{Y}_{2_B2_A}^{AJB}$, $\mathbf{Y}_{2_B2_B}^{AJB}$. The derivation involves a combination of matrix/equation manipulations, matrix inversion operations (as detailed in appendix A), and the application of the system's symmetry properties.

The starting point is isolating $\mathbf{Y}^{\text{tot}}$ from eq. (15a):

$$\mathbf{Y}^{\text{tot}} = -\left(\mathbf{Y}^{A}\right)^{-1}\left(\mathbf{Y}_{2^A2^A}^{AJB} - \mathbf{Y}^{A}\right)\left(\mathbf{Y}^{A}\right)^{-1} \tag{16}$$

Then, eq. (16) is substituted into eq. (15b), eq. (15c) and eq. (15d) to remove the dependency on $\mathbf{Y}^{\text{tot}}$:

$$\mathbf{Y}^{AJB}_{2^A2^B} = -\left(\mathbf{Y}^{AJB}_{2^A2^A} - \mathbf{Y}^A\right)\left(\mathbf{Y}^A\right)^{-1}\mathbf{Y}^B \tag{17a}$$

$$\mathbf{Y}^{AJB}_{2^B2^A} = -\mathbf{Y}^B\left(\mathbf{Y}^A\right)^{-1}\left(\mathbf{Y}^{AJB}_{2^A2^A} - \mathbf{Y}^A\right) \tag{17b}$$

$$\mathbf{Y}^{AJB}_{2^B2^A} = \mathbf{Y}^B + \mathbf{Y}^B\left(\mathbf{Y}^A\right)^{-1}\left(\mathbf{Y}^{AJB}_{2^A2^A} - \mathbf{Y}^A\right)\left(\mathbf{Y}^A\right)^{-1}\mathbf{Y}^B \tag{17c}$$

The next step involves isolating $\mathbf{Y}^B$ from eq. (17a):

$$\mathbf{Y}^B = -\mathbf{Y}^A\left(\mathbf{Y}^{AJB}_{2^A2^A} - \mathbf{Y}^A\right)^{-1}\mathbf{Y}^{AJB}_{2^A2^B} \tag{18}$$

Further, by substituting eq. (17b) into eq. (17a) to eliminate $\mathbf{Y}^B$, the following is obtained:

$$\left(\mathbf{Y}^{AJB}_{2^A2^B}\right)^{-1} = \left(\mathbf{Y}^A\right)^{-1}\left(\mathbf{Y}^{AJB}_{2^A2^A} - \mathbf{Y}^A\right)\left(\mathbf{Y}^{AJB}_{2^B2^A}\right)^{-1}\mathbf{Y}^A\left(\mathbf{Y}^{AJB}_{2^A2^A} - \mathbf{Y}^A\right)^{-1} \tag{19}$$

Then, eq. (18) is substituted into eq. (17c):

$$\mathbf{Y}^{AJB}_{2^B2^B} = \left(\left(\mathbf{Y}^{AJB}_{2^A2^B}\right)^{-1}\left(\mathbf{Y}^{AJB}_{2^A2^A}\left(\mathbf{Y}^A\right)^{-1} - \mathbf{I}\right)\right)^{-1}\left(\left(\mathbf{Y}^A\right)^{-1}\mathbf{Y}^{AJB}_{2^A2^B} - \mathbf{I}\right) \tag{20}$$

This can be rearranged as:

$$\left(\mathbf{Y}^{AJB}_{2^A2^B}\right)^{-1}\left(\mathbf{Y}^{AJB}_{2^A2^A}\left(\mathbf{Y}^A\right)^{-1} - \mathbf{I}\right)\mathbf{Y}^{AJB}_{2^B2^B} = \left(\mathbf{Y}^A\right)^{-1}\mathbf{Y}^{AJB}_{2^A2^B} - \mathbf{I} \tag{21}$$

Next, eq. (19) is substituted into eq. (21):

$$\left(\mathbf{Y}^A\right)^{-1}\mathbf{Y}^{AJB}_{2^A2^A}\left(\mathbf{Y}^{AJB}_{2^B2^A}\right)^{-1}\mathbf{Y}^{AJB}_{2^B2^B} - \left(\mathbf{Y}^{AJB}_{2^B2^A}\right)^{-1}\mathbf{Y}^{AJB}_{2^B2^B} = \left(\mathbf{Y}^A\right)^{-1}\mathbf{Y}^{AJB}_{2^A2^B} - \mathbf{I} \tag{22}$$

Finally, $\mathbf{Y}^A$ can be isolated:

$$\mathbf{Y}^A = \left(\mathbf{Y}^{AJB}_{2^A2^A}\left(\mathbf{Y}^{AJB}_{2^B2^A}\right)^{-1}\mathbf{Y}^{AJB}_{2^B2^B} - \mathbf{Y}^{AJB}_{2^A2^B}\right)\left(\mathbf{Y}^{AJB}_{2^B2^B} - \mathbf{Y}^{AJB}_{2^B2^A}\right)^{-1}\mathbf{Y}^{AJB}_{2^B2^A} \tag{23}$$

By leveraging the symmetry of the admittance and its inverse (impedance), eq. (23) can be rewritten as:

$$\mathbf{Y}^A = \mathbf{Y}^{AJB}_{2^A2^B}\left(\mathbf{Y}^{AJB}_{2^B2^B} - \mathbf{Y}^{AJB}_{2^A2^B}\right)^{-1}\left(\mathbf{Y}^{AJB}_{2^A2^A}\left(\mathbf{Y}^{AJB}_{2^B2^A}\right)^{-1}\mathbf{Y}^{AJB}_{2^B2^B} - \mathbf{Y}^{AJB}_{2^A2^B}\right) \tag{24}$$

This result is equivalent to the primal derivation in eq. (12a).
In analogy to the above derivation, by mirroring the operations with respect to $\mathbf{Y}^B$, the equivalent of eq. (12b) is obtained:

$$\mathbf{Y}^B = \left(\mathbf{Y}^{AJB}_{2^A2^A}\left(\mathbf{Y}^{AJB}_{2^B2^A}\right)^{-1}\mathbf{Y}^{AJB}_{2^B2^B} - \mathbf{Y}^{AJB}_{2^A2^B}\right)\left(\mathbf{Y}^{AJB}_{2^A2^A} - \mathbf{Y}^{AJB}_{2^A2^B}\right)^{-1}\mathbf{Y}^{AJB}_{2^A2^B} \tag{25}$$

Finally, manipulating eq. (15b) to isolate the joint admittance yields:

$$\mathbf{Y}^J_c = \mathbf{Y}^B\left(\mathbf{Y}^{AJB}_{2^A2^B}\right)^{-1}\mathbf{Y}^A - \mathbf{Y}^A - \mathbf{Y}^B \tag{26}$$

where $\mathbf{Y}^A$ and $\mathbf{Y}^B$ can be expressed as functions of the assembled system (see eq. (24) and eq. (25)). Consequently, the isolated joint must also be a function of the assembled system dynamics alone. Given the complexity of deriving the simplified form of eq. (26) via substitution of eq. (24) and eq. (25), a different approach will be employed to complete the proof of equivalence with the primal expression in eq. (12c) (or eq. (13)).

A proof by verification is used, where the ansatz solution in eq. (12c) is substituted into eq. (26) to confirm that it satisfies the system.

First, $\mathbf{Y}^A$ from eq. (23) (equivalent to eq. (24)) and $\mathbf{Y}^B$ from eq. (25) are rewritten using the ansatz for the joint admittance (eq. (12c)) and exploiting the symmetry of the cross-coupling terms of the assembled admittance:

$$\mathbf{Y}^A = \mathbf{Y}^J_c\left(\mathbf{Y}^{AJB}_{2^B2^B} - \left(\mathbf{Y}^{AJB}_{2^A2^B}\right)^T\right)^{-1}\left(\mathbf{Y}^{AJB}_{2^A2^B}\right)^T \tag{27a}$$

$$\mathbf{Y}^B = \mathbf{Y}^J_c\left(\mathbf{Y}^{AJB}_{2^A2^A} - \mathbf{Y}^{AJB}_{2^A2^B}\right)^{-1}\mathbf{Y}^{AJB}_{2^A2^B} \tag{27b}$$

Substituting eq. (27a) and eq. (27b) into eq. (26) results in:

$$\begin{aligned}\mathbf{Y}_c^J = \mathbf{Y}_c^J \left(\mathbf{Y}_{2_A2_A}^{AJB} - \mathbf{Y}_{2_A2_B}^{AJB}\right)^{-1} \mathbf{Y}_{2_A2_B}^{AJB} \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^{-1} \mathbf{Y}_c^J \left(\mathbf{Y}_{2_B2_B}^{AJB} - \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T\right)^{-1} \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T \\ - \mathbf{Y}_c^J \left(\mathbf{Y}_{2_B2_B}^{AJB} - \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T\right)^{-1} \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T - \mathbf{Y}_c^J \left(\mathbf{Y}_{2_A2_A}^{AJB} - \mathbf{Y}_{2_A2_B}^{AJB}\right)^{-1} \mathbf{Y}_{2_A2_B}^{AJB}\end{aligned} \tag{28}$$

After cancellation of redundant terms:

$$\begin{aligned}\mathbf{I} = \left(\mathbf{Y}_{2_A2_A}^{AJB} - \mathbf{Y}_{2_A2_B}^{AJB}\right)^{-1} \mathbf{Y}_c^J \left(\mathbf{Y}_{2_B2_B}^{AJB} - \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T\right)^{-1} \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T \\ - \left(\mathbf{Y}_{2_B2_B}^{AJB} - \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T\right)^{-1} \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T - \left(\mathbf{Y}_{2_A2_A}^{AJB} - \mathbf{Y}_{2_A2_B}^{AJB}\right)^{-1} \mathbf{Y}_{2_A2_B}^{AJB}\end{aligned} \tag{29}$$

By further elimination:

$$\begin{aligned}\left(\mathbf{Y}_{2_A2_A}^{AJB} - \mathbf{Y}_{2_A2_B}^{AJB}\right) = \mathbf{Y}_c^J \left(\mathbf{Y}_{2_B2_B}^{AJB} - \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T\right)^{-1} \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T \\ - \left(\mathbf{Y}_{2_A2_A}^{AJB} - \mathbf{Y}_{2_A2_B}^{AJB}\right) \left(\mathbf{Y}_{2_B2_B}^{AJB} - \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T\right)^{-1} \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T - \mathbf{Y}_{2_A2_B}^{AJB}\end{aligned} \tag{30}$$

Further simplification results in:

$$\mathbf{Y}_{2_A2_A}^{AJB} \left(\left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T\right)^{-1} \left(\mathbf{Y}_{2_B2_B}^{AJB} - \left(\mathbf{Y}_{2_A2_B}^{AJB}\right)^T\right) = \mathbf{Y}_c^J - \left(\mathbf{Y}_{2_A2_A}^{AJB} - \mathbf{Y}_{2_A2_B}^{AJB}\right) \tag{31}$$

This leads to the final simplified form:

$$\mathbf{Y}_c^J = \mathbf{Y}_{2_A2_A}^{AJB} \left(\mathbf{Y}_{2_B2_A}^{AJB}\right)^{-1} \mathbf{Y}_{2_B2_B}^{AJB} - \mathbf{Y}_{2_A2_B}^{AJB} \tag{32}$$

By substituting the ansatz solution, this is reduced to the trivial identity:

$$\mathbf{Y}_c^J = \mathbf{Y}_c^J \tag{33}$$

This concludes the verification.

As a remark, the results in eq. (24), eq. (25) and eq. (32) are not unique. Equivalent expressions can be obtained by reordering or rearranging the equations, applying matrix inversion properties, and leveraging the symmetry inherent in the matrices. This has particular significance from an experimental perspective for two key reasons:

- Measurement errors may propagate through the inversion operations. As a result, the choice of the mathematical expression used for identification could impact the accuracy of the outcome.
- While the primary goal of the derivation in section 3.2 is to obtain an 'in-situ' expression for the joint admittance, alternative formulations for joint isolation, based on the dynamics of the assembled system and its subcomponents, can be readily derived by inverting the equations in eq. (15). These alternative expressions may be advantageous in certain scenarios where access to component locations is limited.

Conclusion

This paper presents a concise review of popular substructuring quasi-static joint identification methods, focusing on the primal impedance-based and dual admittance-based formulations. The assumptions underlying both approaches are clearly outlined, with attention to their significance and the specific stages of the derivations where they are introduced. The analysis explores the equivalence of the two formulations and revisits the dual admittance-based derivation to highlight key aspects of isolating joint and subcomponents 'in-situ' within the admittance domain.

Ongoing work focuses on generalizing the quasi-static identification framework to incorporate non-interface measurements, aiming to further enhance the methodology.

Some Matrix Inversion Properties

Some relevant matrix inversion properties used for the derivations in section 3.1 and section 3.2 are provided in the following:

$$\mathbf{S} := \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}, \quad \mathbf{S}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{bmatrix} \tag{34}$$

$$\left(\mathbf{A}^{-1} + \mathbf{B}^{-1}\right)^{-1} = \mathbf{A}\left(\mathbf{A} + \mathbf{B}\right)^{-1}\mathbf{B} \tag{35}$$

$$\left(\mathbf{A}\mathbf{B}\mathbf{C}\right)^{-1} = \mathbf{C}^{-1}\mathbf{B}^{-1}\mathbf{A}^{-1} \tag{36}$$

$$\left(\mathbf{A}\mathbf{B}\right)^{-1} = \mathbf{B}^{-1}\mathbf{A}^{-1} \tag{37}$$

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