Chapter 7

Frequency Response Function Expansion using a Symmetry Preserving Gysin Expansion Technique

Christopher Page and Peter Avitabile

Abstract The theoretical basis for frequency based substructuring (FBS) has been established for decades. Despite this, using FBS with measured data is extremely challenging due to measurement errors and practical limitations related to measuring rotational degrees-of-freedom (r-DoF). This paper revisits a previously proposed approach, known as Gysin expansion, that models the response using columns of an impedance matrix from a finite element model (FEM), expanding measured FRFs in a least squares sense. The original technique is shown to be robust with respect to FEM errors, but the expanded FRFs lack reciprocity. FRF reciprocity implies a passive structural system, which is an important characteristic for FBS applications. In this work, a novel extension of Gysin expansion is presented that ensures the expanded FRFs are reciprocal (symmetric) and thus describe a passive structural system. The extended method is also shown to be robust with respect to modeling errors and can be used to estimate unmeasured r-DoF. Moreover, the technique can reconstruct an unmeasured drive-point partition of an FRF matrix using measured cross-FRFs only. Because experimental techniques used to acquire drive-point measurements often introduce bias errors that render FBS predictions functionally useless, this new approach to FRF expansion helps mitigate the effects of some measurement errors on FBS. An Example using numerically derived data is presented. Similarities to other techniques, such as System Equivalent Modeling Mixing (SEMM) are highlighted.

Keywords Gysin expansion · Reciprocity · Expansion techniques · Rotational DoF · Frequency based substructuring

Introduction

Frequency based substructuring requires frequency response functions (FRFs) at substructure interfaces that are often inaccessible or unmeasurable using conventional testing methodologies. Techniques for FRF expansion include the dynamic expansion [1], the virtual point transformation (VPT) [2] and system equivalent mixing model (SEMM) [3, 4] amongst others. However, these expansion techniques result in expanded FRFs which violate important inductive biases from modal theory, e.g., reciprocity, passivity or minimality (i.e., rank-1 modal residues). This paper presents an enhanced FRF expansion methodology which produces reciprocal FRFs [5]. The methodology is applied to FRFs expanded using a formulation of dynamic expansion known as Gysin expansion [6] which is more robust to modeling errors between the FEM derived basis functions and experimental data. The paper highlights that the long established Gysin expansion technique is identical the more recently proposed SEMM.

FRF Expansion Methodologies

Dynamic expansion is an expansion technique that relies on basis functions derived from a FEM's dynamic stiffness matrix. The expansion may be expressed in an impedance or admittance form. The impedance form is derived by partitioning the

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FEM's dynamic stiffness matrix along a set of, i, interior DoF and b, boundary/measured DoF as

$$\begin{bmatrix}\mathbf{Z}_{ii} & \mathbf{Z}_{ib}\\ \mathbf{Z}_{bi} & \mathbf{Z}_{bb}\end{bmatrix}\begin{bmatrix}u_i\\ u_b\end{bmatrix}=\begin{bmatrix}f_i\\ f_b\end{bmatrix} \tag{1}$$

where u_i represents the displacements at the interior DoF, u_b the displacements at the boundary DoF, f_i are the external forces on the interior DoF and f_b are the external forces acting on the boundary DoF. The impedance form of dynamic expansion expresses the displacement at the interior DoF in terms of the external forces on the interior DoF and the measured/boundary DoF according to

$$u_i = \mathbf{Z}_{ii}^{-1} f_i - \mathbf{Z}_{ii}^{-1}\mathbf{Z}_{ib} u_b \tag{2}$$

By leveraging block matrix inversion the admittance form of dynamic expansion can be shown to be

$$u_i = \mathbf{Y}_{ii} - \mathbf{Y}_{ib}\mathbf{Y}_{bb}^{-1}\left(\mathbf{Y}_{bb}-\hat{\mathbf{Y}}_{bb}\right)\mathbf{Y}_{bb}^{-1}\mathbf{Y}_{bi} f_i + \mathbf{Y}_{ib}\mathbf{Y}_{bb}^{-1}\hat{\mathbf{Y}}_{bb} f_b \tag{3}$$

and by extension the entire expanded FRF matrix in block format becomes

$$\mathbf{Y}^{(\text{expanded})} = \begin{bmatrix}\mathbf{Y}_{ii} - \mathbf{Y}_{ib}\mathbf{Y}_{bb}^{-1}\left(\mathbf{Y}_{bb}-\hat{\mathbf{Y}}_{bb}\right)\mathbf{Y}_{bb}^{-1}\mathbf{Y}_{bi} & \mathbf{Y}_{ib}\mathbf{Y}_{bb}^{-1}\hat{\mathbf{Y}}_{bb}\\ \hat{\mathbf{Y}}_{bb}\mathbf{Y}_{bb}^{-1}\mathbf{Y}_{bi} & \hat{\mathbf{Y}}_{bb}\end{bmatrix} \tag{4}$$

where in (3) and (4), $\hat{\mathbf{Y}_{bb}}$ indicates admittance functions that come from measurements with all other admittance functions synthesized from the FEM. The admittance form of dynamic expansion is equivalent to SEMM with a conforming interface [3].

Equation (4) illustrates a particular defect with dynamic expansion. Namely when there are inconsistencies between the measured admittance functions and the FEM synthesized FRFs at the boundary DoF, the boundary partition admittance functions do not cancel each other out, producing spurious peaks in the FRFs at resonant frequencies of the FEM constrained at the boundary DoF. In the cross-FRFs imperfect pole cancellation in the term $\hat{\mathbf{Y}}_{bb}\mathbf{Y}_{bb}^{-1}$ has the same result. When $\hat{\mathbf{Y}}_{bb} = \mathbf{Y}_{bb}$ the dynamic expansion process is exact. The defect will be illustrated using an analytical case study later in the paper.

Gysin Expansion

Partitioning the columns of the finite element model dynamic stiffness matrix along interior and boundary DoF as

$$\begin{bmatrix}\mathbf{Z}_{ni} & \mathbf{Z}_{nb}\end{bmatrix}\begin{bmatrix}u_i\\ u_b\end{bmatrix}=\begin{bmatrix}f_i\\ f_b\end{bmatrix} \tag{5}$$

Gysin expansion estimates the response at the interior DoF by

$$u_i = \mathbf{Z}_{ni}^{\dagger}\begin{bmatrix}f_i\\ f_b\end{bmatrix} - \mathbf{Z}_{ni}^{\dagger}\mathbf{Z}_{nb} u_b \tag{6}$$

where † denotes the generalized inverse. Functionally, Gysin expansion is similar to dynamic expansion. However, all n finite element DoF are used in the expansion, therefore any error between the tested structure and finite element model is distributed across all FEM DoF. In this way, Gysin expansion avoids the defects of dynamic expansion when the finite element model and measured data are inconsistent. Gysin expansion is the impedance equivalent to SEMM over an extended interface [4].

Symmetry Constrained Gysin Expansion

A formulation of Gysin expansion which respects reciprocity is obtained by recognizing the expansion process as a linear regression problem in the columns of the FEM's dynamic stiffness matrix associated with the interior DoF. Modifying (5) such that the interior DoF are replaced by the FRFs between interior DoF, the forces on the interior DoF are replaced by an identity matrix, the forces on the boundary DoF by a null matrix and the boundary DoF response by the cross-FRFs between the interior and boundary DoF gives

$$\mathbf{Z}_{ni}\mathbf{Y}_{ii} = \begin{bmatrix}\mathbf{I}_i\\ 0\end{bmatrix} - \mathbf{Z}_{nb}\hat{\mathbf{Y}}_{bi} \tag{7}$$

Introducing a symmetry constraint for the form

$$\mathbf{B}\text{vec}\left(\mathbf{Y}_{ii}\right) = 0 \tag{8}$$

where $\mathbf{B}$ is a signed Boolean matrix that "picks-out" those entries in $\mathbf{Y}_{ii}$ that are related by symmetry, i.e. $\mathbf{Y}_{kj} - \mathbf{Y}_{jk} = 0$, q and combining the constraint with a vectorized form of (7) produces a constrained linear regression problem of the form

$$\begin{bmatrix} \left(\mathbf{I}_i \otimes \mathbf{Z}_{ni}^T \mathbf{Z}_{ni}\right) & \mathbf{B}^T \\ \mathbf{B} & 0 \end{bmatrix} \begin{bmatrix} \text{vec}\left(\mathbf{Y}_{ii}\right) \\ \lambda \end{bmatrix} = \begin{bmatrix} \text{vec}\left(\mathbf{Z}_{ni}^T \begin{bmatrix} \mathbf{I}_i \\ 0 \end{bmatrix} - \mathbf{Z}_{ni}^T \mathbf{Z}_{nb} \hat{\mathbf{Y}}_{bi}\right) \\ 0 \end{bmatrix} \tag{9}$$

In (9), λ is a vector of Lagrange multipliers associated with the symmetry constraint. Solving (9) for the Lagrange multipliers, the FRFs expanded to the interior DoF can be shown to be

$$\begin{aligned} \text{vec}\left(\mathbf{Y}_{ii}\right) = \text{vec}\left(\mathbf{Y}_{ii}^0\right) - \\ \left(\mathbf{I}_i \otimes \left(\mathbf{Z}_{ni}^T \mathbf{Z}_{ni}\right)^{-1}\right) \mathbf{B}^T \left[\mathbf{B}\left(\mathbf{I}_i \otimes \left(\mathbf{Z}_{ni}^T \mathbf{Z}_{ni}\right)^{-1}\right) \mathbf{B}^T\right]^{-1} \mathbf{B}\text{vec}\left(\mathbf{Y}_{ii}^0\right) \end{aligned} \tag{10}$$

where $\mathbf{Y}_{ii}^0$ represents the Gysin expanded FRFs with no reciprocity constraint.

Numerical Case Study

In this section the defects of conventional dynamic expansion are demonstrated using data synthetically generated from two FEMs. Standard Gysin expansion is shown to eliminate the defects of dynamic expansion but produce expanded FRFs that are not reciprocal. Finally the symmetry constrained Gysin expansion process is applied resulting in reciprocal FRFs.

Figure (1) presents the finite element models used to generated data for this case study. The model on the left was used to generate the measured data. The model on the right was used for dynamic expansion. Modeling errors were introduced by using two different physics models for beam bending, perturbing material properties and using different boundary conditions for the 'test' finite element model and 'expansion' model. These differences are summarized in the table in Figure (1).

Table: Example of Errors Between Tested and Modeled Structure

Model	Physics	Length [in.]	Width [in.]	Thickness [in.]	Elastic Modulus [psi]	Density [lb/in³]	Boundary Rotary Stiffness [lbf-in/rad]	Boundary Rotary Stiffness [lbf/in]
Test	Timoshenko	35.8125	2.85	1.02	11E6	0.103	1E4 (left) /9.87E3 (right)	1.17E3 (left) /9.3E3 (right)
FEM	Euler-Bernoulli	36	3	1	10E6	0.1	0 (left) / 0 (right)	1E3 (left) / 1E3 (right)

Fig. 1 Beam Model for Gysin Expansion Study

The test finite element model is used to simulated FRFs at four response points due to inputs at three reference DoF. The response measurements are indicated by the green squares in Figure (1) and the input points are marked using the impact hammer. No drive-point FRFs are acquired. Dynamic expansion is be used to estimate the drive-point partition at the reference DoF. Figure (2) compares the dynamically expanded FRFs at the reference DoF (dashed orange) with the ground truth FRFs obtained from the test model. Despite the modeling error, the dynamically expanded FRFs are accurate up to 200 Hz. However, two prominent spurious resonances at 591 and 705 Hz affect the accuracy of the expansion from 200-1000 Hz for most FRFs. Each of these spurious resonances are associated with natural frequencies of the 'FEM' model constrained at those DoF co-located with the response measurements. Figure (3) shows the same FRFs expanded using Gysin expansion. By using Gysin expansion, the spurious resonances have been eliminated completely and the expanded FRFs have excellent agreement with the ground truth FRFs. However, Figure (4) compares expanded cross-FRFs illustrating that these FRFs are not reciprocal.

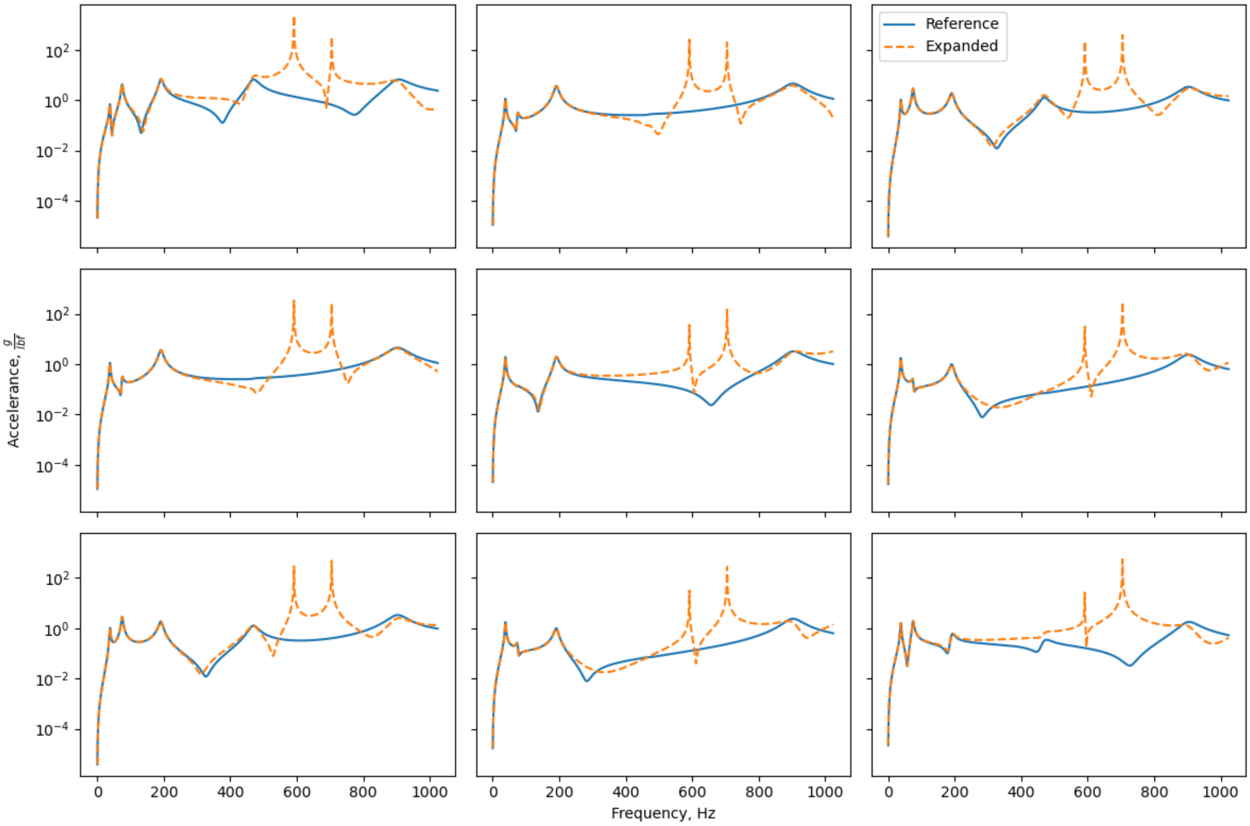

Fig. 2 Defects of Dynamic Expansion

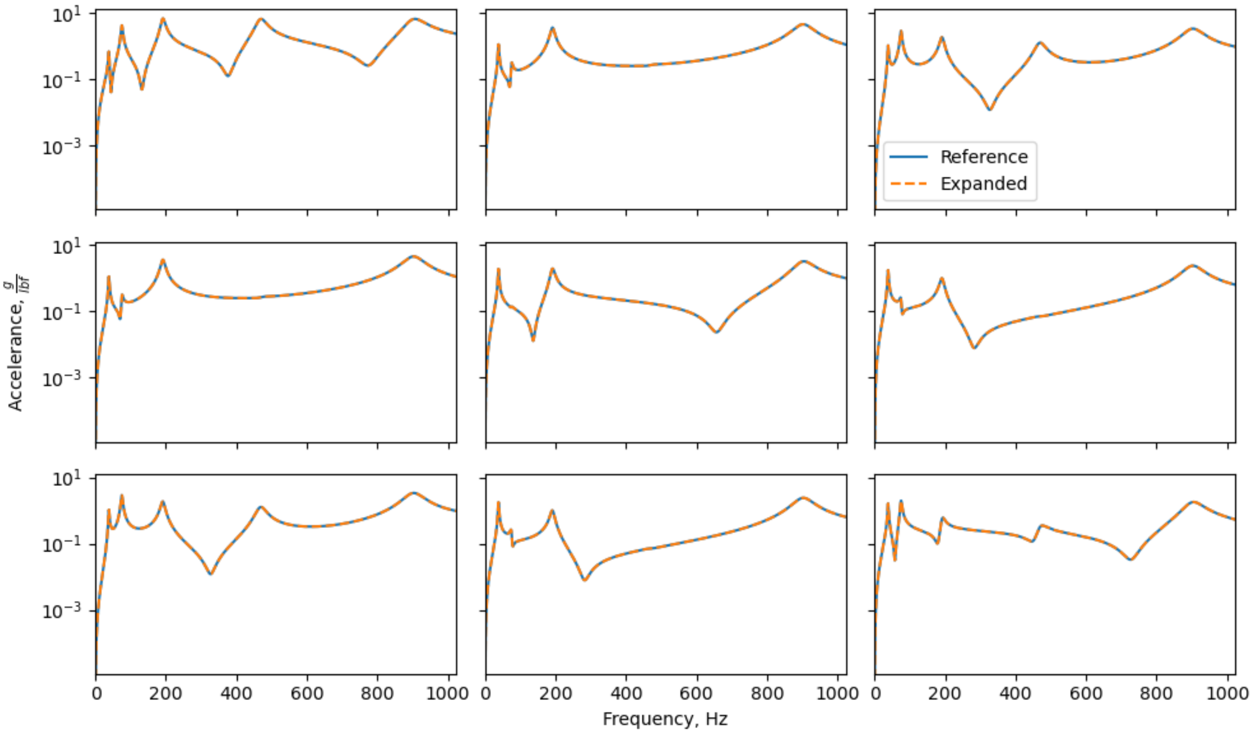

Fig. 3 Gysin Expanded FRFs

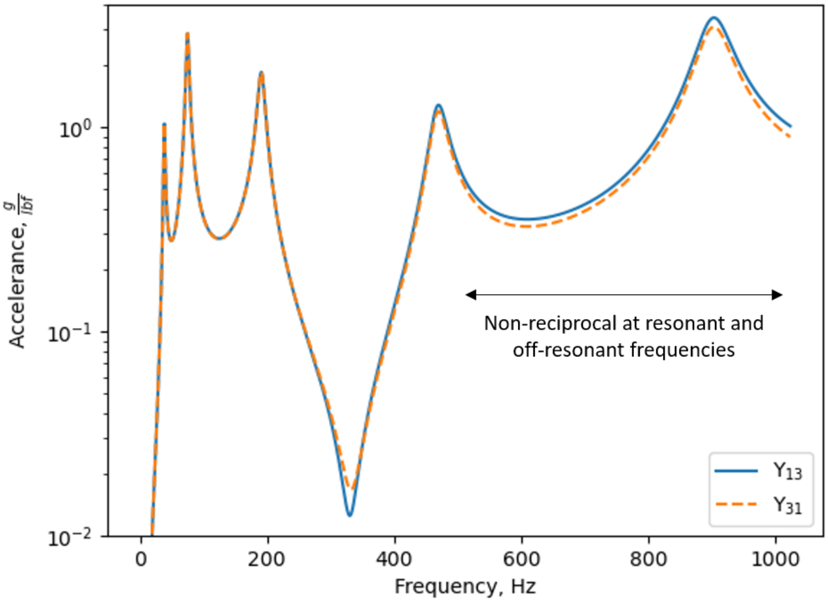

Fig. 4 Lack of Reciprocity in Gysin Expanded FRFs

The expanded drive point partition is presented in Figure (5) and Figure (6) shows two reciprocal cross-FRFs from expanded drive point partition illustrating that the symmetry constrained was observed through the expansion process.

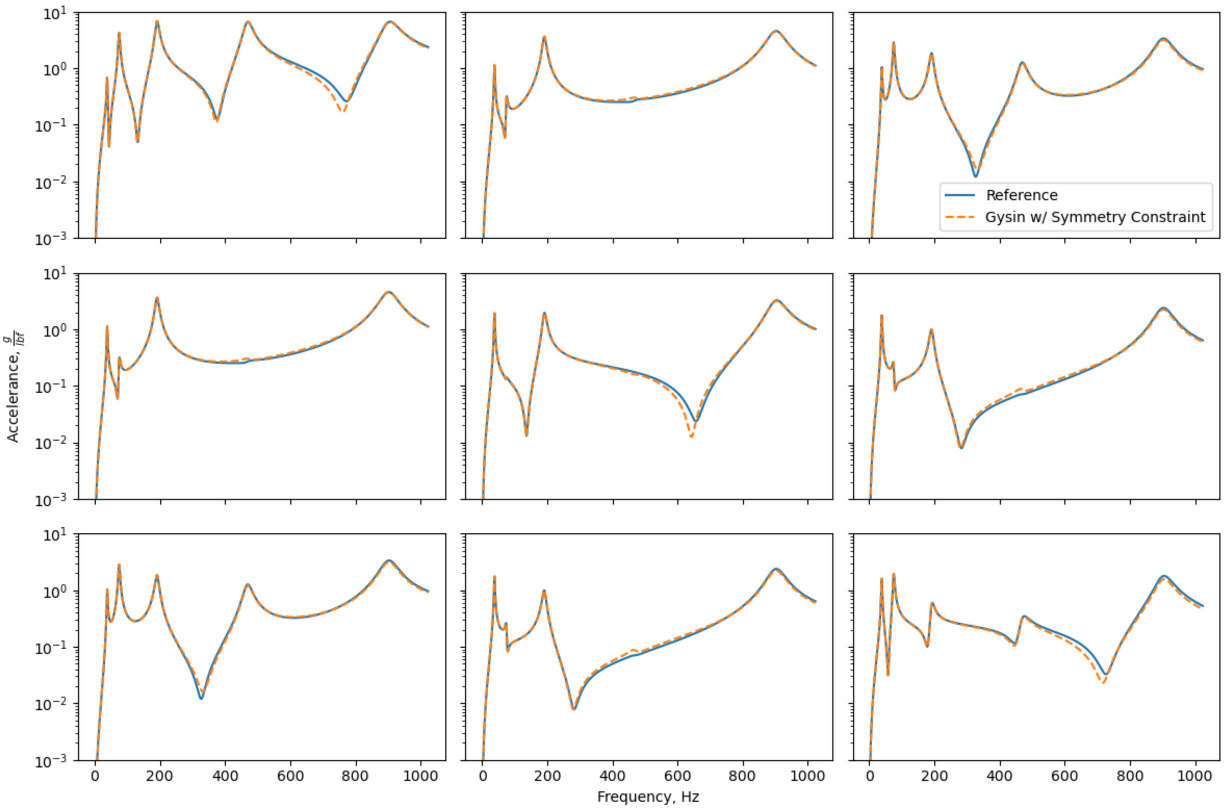

Fig. 5 Gysin Expanded FRFs with Reciprocity Constraint

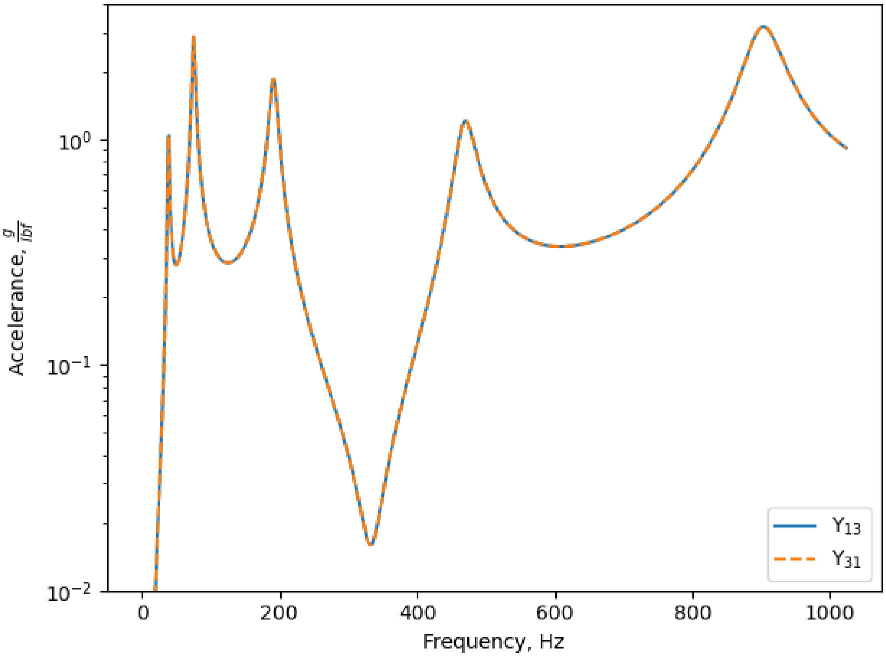

Fig. 6 Gysin Expanded FRFs with Reciprocity Constraint; Comparison of Reciprocal Measurements

Conclusion

In this paper, Gysin expansion was shown to eliminate defects with dynamic expansion when there are inconsistencies between the FEM basis functions and measured data. However, the Gysin expanded FRFs are not reciprocal. The Gysin expansion process was re-cast as a linear regression problem and a symmetry constraint was introduced to ensure expanded FRFs maintain reciprocity. The methodology was demonstrated using data synthetically generated from finite element models.

References

1. Paz, M. “Dynamic condensation”. *AIAA Journal*, 23(5) (1985)
2. Van der Seijs, M., van den Bosch, D., Rixen, D., and Klerk, D. “An improved methodology for the virtual point transformation of measured frequency response functions in dynamic substructuring”. In *Proceedings of COMPDYN 2013*, pages 4334–4347 (2013)
3. Klaassen, S.W.B. and Seijs, M.V.v.d. “Introducing SEMM: A novel method for hybrid modelling”. In Linderholt, A., Allen, M.S., Mayes, R.L., and Rixen, D., editors, *Dynamics of Coupled Structures, Volume 4*, Conference Proceedings of the Society for Experimental Mechanics Series, pages 117–125. Springer International Publishing (2018)
4. Klaassen, S.W.B., van der Seijs, M.V., and de Klerk, D. “System equivalent model mixing”. *Mechanical Systems and Signal Processing*, 105:90–112 (2018)
5. Page, C. *A Unified Substructuring Approach to Fixture Neutralization*. Phd thesis, University of Massachusetts - Lowell, Lowell, CA (2020)
6. Gysin, H. “Comparison of expansion method for FE modeling error localization”. In *Proceedings of the 8th International Modal Analysis Conference* (1990)