Chapter 10
Motivating Multivariate Specifications for Multiple-Input, Multiple-Output Vibration Testing

Colin Haynes, Thomas Thompson, and Adam Watts

Abstract The concept of a conservative, bounding environment for multiple-input, multiple-output (MIMO) vibration testing is significantly more complex than in the traditional univariate case. In the univariate case, one-sided tolerance bounds provide a convenient definition of a conservative environment, expressing the level of conservatism in terms of coverage and confidence. Recent work has shown how the same concept cannot be applied directly to the multivariate case. However, the case of MIMO control is necessarily a multivariate one because the target responses are no longer considered independently. Rather than attempt to apply univariate methods to the MIMO case, this paper argues for adapting the definition of a conservative environment to better reflect the reality of the service environment.

Keywords Multivariate · Specifications · MIMO · Environmental Testing · Conservatism

Introduction

For many decades, electrodynamic shakers have been used to test aerospace hardware under simulated random vibration environments. Most such laboratory environmental testing is conducted on one axis at a time, though recently multi-axis methods have the subject of much study [1–4]. MIMO testing holds much promise for more accurately simulating the inherently multi-axial nature of most service environments. However, while established methods exist for developing specifications for single-axis testing, no consensus has been achieved yet regarding the appropriate methods for MIMO specifications. Other authors have noted the difficulty of determining MIMO specifications, but none to the authors' knowledge have considered the resulting multivariate nature of the statistics [5, 6]. Though there are likely other applications for the multivariate concepts discussed here, the primary focus of this paper is on the development of environmental specifications for MIMO structural dynamics testing.

First, the process for univariate specifications will be outlined briefly. Some multivariate statistical methods will then be examined to determine their suitability for deriving MIMO specifications. The concepts of coverage and confidence are revisited in a multivariate context, and several limitations to extending existing univariate approaches are discussed. Finally, a simple analytical example is used to demonstrate that the relationships between response variates is important to deriving an accurate environmental specification.

Univariate Specifications

Typically, aerospace structures are tested to environments that represent a statistically-conservative envelope of expected service environments to ensure a specified level of reliability in operation. An environmental specification for a given level of conservatism is most often determined by calculating the one-sided normal tolerance bound (or one-sided normal tolerance interval). The assumption for the mechanical testing application is that extreme low responses are not relevant for failure, and only extreme responses on the upper end of the distribution need to be considered in establishing a statistically-conservative specification. Mil-Std-810 (see method 514.8, Annex F and method 517.3) discusses the concept of the maximum predicted environment (MPE), which in that context is defined as a one-sided upper tolerance interval with coverage of 0.95 and

Colin Haynes · Thomas Thompson · Adam Watts
Los Alamos National Laboratory, Los Alamos, NM 87545
e-mail: cmhaynes@lanl.gov; tnthompson@lanl.gov; acwatts@lanl.gov

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confidence of 0.50 [7]. Lalanne also elaborates on the process used to develop random vibration specifications [8]. Despite how common it is to define conservative levels in this way, it is still easy to misinterpret the meaning of such specifications [9].

Furthermore, this approach assumes that mechanical shock and vibration are defined as spectral quantities where each frequency line may be treated as its own independent distribution. Because single-axis testing is by far the most common experimental method, the specification development approach can assume a univariate distribution at each frequency line.

However, when moving to a MIMO test, it may be unwise to ignore the correlations between the several target responses (i.e., the spatial correlation). In general, for a distributed random vibration excitation of a structure, higher levels at one point would generally be correlated with higher responses at other points. However, this basic trend will be modified by the modes of the structure, as at certain frequencies some points on the structure will tend to higher responses than others based on mode shapes.

Figure 1 shows a standard normal distribution plot. Coverage in the univariate space is synonymous with the cumulative distribution function (CDF). Assuming a Gaussian distribution, the desired coverage is the area under the probability density function (PDF) curve. To establish that, one can perform and inverse CDF function on the desired coverage to obtain the data value that bounds the distribution. Total area under the PDF curve is equal to 1. Unfortunately, this is not the case in the multivariate space.

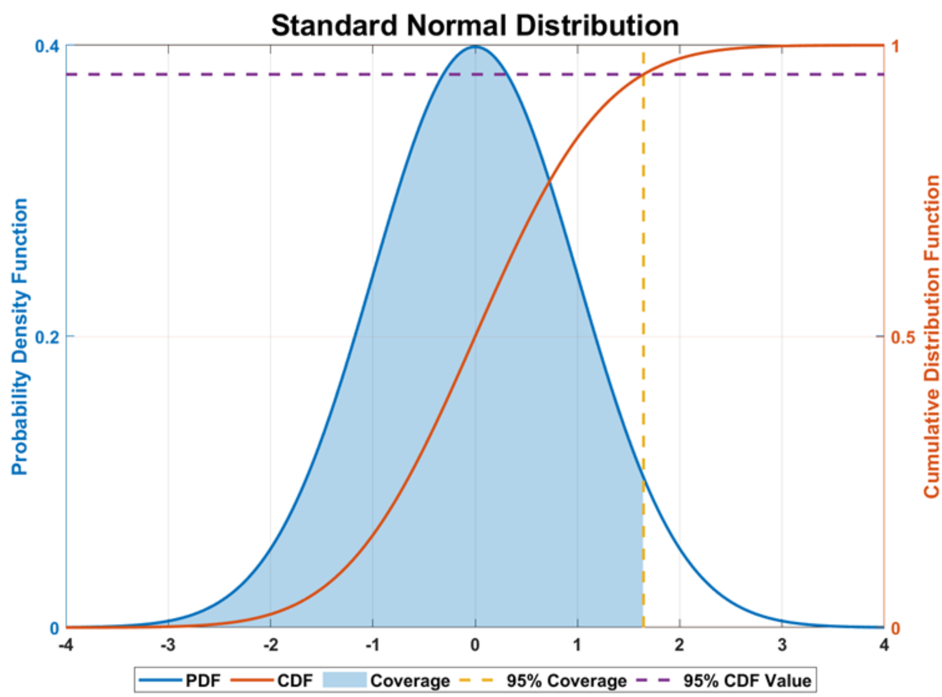

Fig. 1 Examples of PDF and CDF levels in the univariate case.

Simultaneous Prediction Intervals

Perhaps the clearest extension of the traditional univariate specification development approach to the multivariate case is the simultaneous prediction interval. In this case, the specification is based on meeting the defined coverage and confidence level on each variate independently, simply adjusting the confidence parameter by a Bonferroni correction that accounts for the number of variates and ensures that each variate is covered at the specified level [10]. While extremely efficient to compute, the Bonferroni correction does not consider any correlation between variates. Worse, the actual conservatism of the specification when considered relative to a known multivariate distribution increases as the number of variates increases.

Thus, although using more control DOF would generally improve match between a MIMO test and the field test reality, the simultaneous prediction interval would penalize using more control degrees of freedom by continuing to ratchet up the conservatism of the specification with every added variate. In this way, SISO tests would be specified at the lowest level of conservatism, despite yielding (in general) the poorest average match to the desired environment across the entire structure. We believe that a more sophisticated specification approach is required to take advantage of the potential of MIMO testing.

Dimensionality of Multivariate Specifications

Univariate tolerance bounds for a particular coverage proportion, usually denoted as β, are simply scalar. Multivariate analogies are necessarily of a higher dimension. Recall that one-sided tolerance bounds are equivalent to a one-sided confidence interval for a normal distribution quantile, denoted as $Q(\tau)$ (Meeker et al. 2023). Where $Q(\bullet)$ is the quantile function and τ is the quantile probability. Denoting the CDF of a random variable X, as $F_X(x)$, the quantile function for strictly monotonic distributions can be written as the inverse of the CDF, $Q(\tau) = F_X^{-1}(x)$.

For a univariate one-sided, upper, tolerance bound, has a particular case where $\beta = \tau$, i.e., the one-sided proportion of the population equals the quantile probability. The solutions for the intersection of the tolerance bounds and the quantile functions are one dimension less than the CDF for the respective distribution. For the normal univariate case, the CDF is a curve with the following mapping $F : \mathbb{R} \rightarrow [0,\ 1]$. Thus, the solution for the quantile function for a particular probability is just a scalar. However, the for the bivariate case, the CDF is a surface with the following mapping $F : \mathbb{R}^2 \rightarrow [0,\ 1]$. Denoting the joint CDF of two random variables X and Y, as $F_{XY}(x, y) = P(X \leq x,\ Y \leq y)$, this implies $Q(\tau) = F_{XY}^{-1}(x, y)$. The solution for the quantile function $Q(\tau)$ for a particular probability is now a curve that is comprised of infinite set of (x, y)vector tuples. Going up in dimensionality to a trivariate distribution yields a CDF that is a volume and the solution for the quantile function $Q(\tau)$ being a surface. For a given target multivariate quantile probability, τ, there exists an infinite set of target responses that satisfy the statistical requirements of being a solution to the quantile function. Methods do exist for determining the hypersurface and selecting reasonable points on that hypersurface that may be most relevant for environmental testing [11].

Furthermore, it should be noted that univariate tests always have a degree of tolerance associated with them to make the test achievable. Tolerancing guidelines for MIMO tests have not yet been established, but current state of the art MIMO practice also tends to allow the cross spectral density terms to vary to find a specification that can be achieved in the lab. Because multiple solutions can be described which satisfy the intent of the statistical conservatism, we suggest that it is best to specify the target environment as a space of solutions, allowing the particular solution to be chosen based on practicality of achieving the specification. Taking advantage of a more flexible specification may be challenging in implementation, but it is not necessary to find the optimal solution. Nominal specifications within the space of acceptable solutions could be prescribed if narrowing a range of choices is difficult *in situ*. Rather, being clear about what aspects of the target environment may be changed while still meeting the testing intent is a benefit in itself. If deviations from a nominal specification become necessary to execute a given test, outlining a space of solutions can provide the experimentalist with more options to perform the most appropriate test possible.

Coverage in a Multivariate Space

Figure 2 shows an example of a bivariate normal PDF and CDF. For simplicity, let us assume a bivariate gaussian distribution with a mean vector of 0 and identity covariance matrix. The PDF is a surface and, in this specific case, constant PDF values would be circles. If the covariance matrix has values in the off-diagonals or if the variances are not unity, the constant PDF values would be ellipses. A CDF curve at a constant value has two asymptotes (one parallel to the x-axis and the other parallel to the y-axis). Additionally, the volume under the PDF surface is equal to 1 (area under the curve for the univariate case and as the variates increase in dimensionality, so does this area/volume metric).

This is where we see a break between the univariate space (where the CDF value and coverage a synonymous) and the multivariate space. In the bivariate example, coverage would be akin to removing sections of the surface and then calculating the volume (which would be less than 1). Assume that a CDF contour is specified as our statistic of choice for designing a specification. In that case, calculating the coverage would involve removing the volume of the PDF curve that is above this CDF curve.

Figure 3 shows a CDF contour of 0.5 where the volume is removed from the PDF surface. The volume of this surface would be considered the coverage. To see the full effects of how the CDF affects coverage, a numerical study was conducted that varied the CDF values along with varying the correlation coefficient between variates 1 and 2.

Figure 4 shows the coverage (with the color map ranging from 0 to 1) as a function of the CDF value (y-axis) and the correlation coefficient (x-axis). The key observation of this plot is that holding a CDF value constant results in different

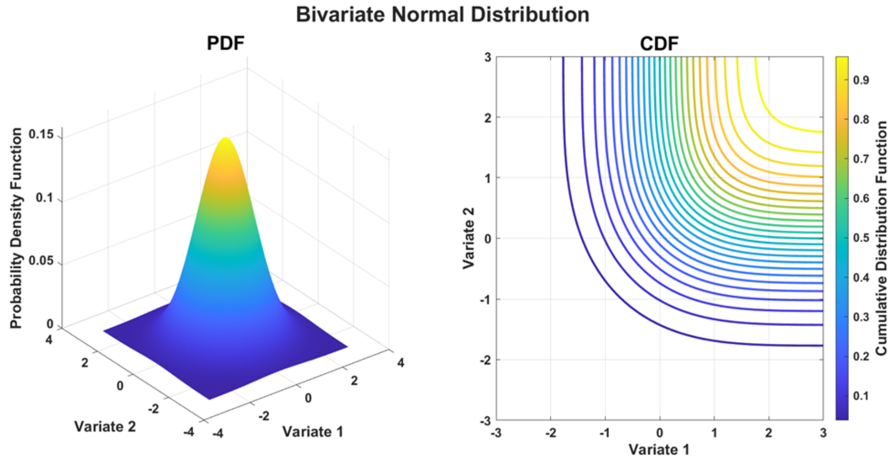

Fig. 2 Example PDF and CDF levels for a bivariate normal distribution.

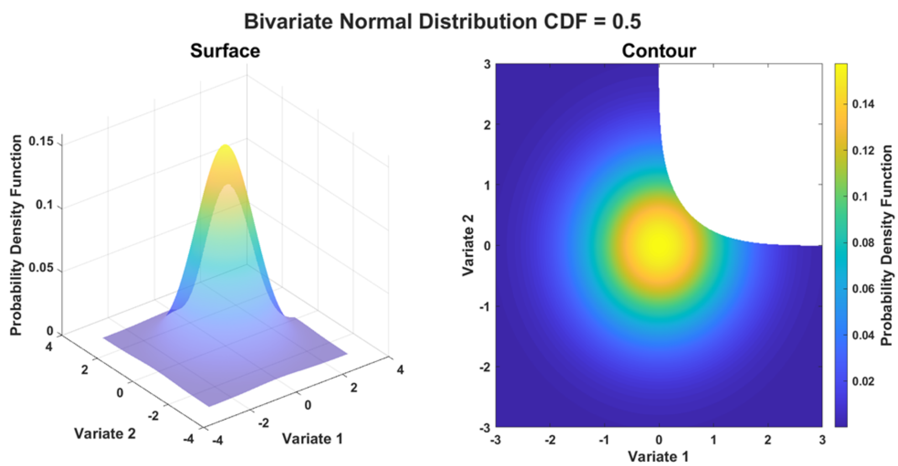

Fig. 3 Example PDF and CDF levels for a bivariate normal distribution, with volume removed at CDF=0.5.

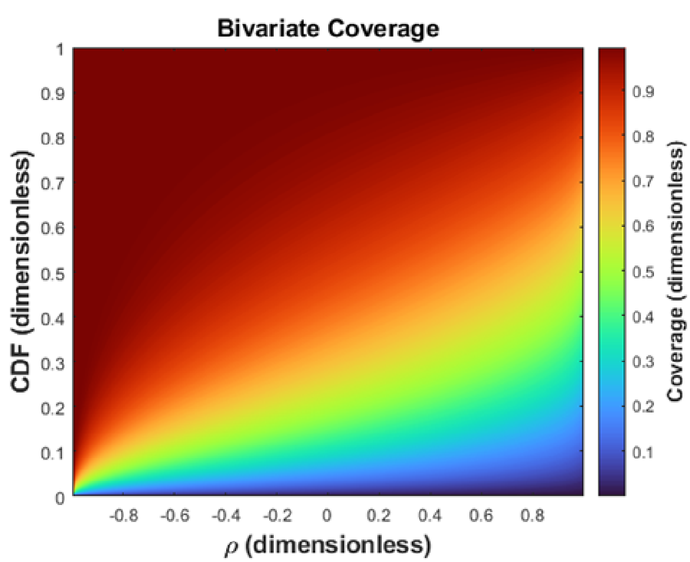

Fig. 4 Coverage as a function of CDF and correlation coefficient.

coverage values based on the correlation of the data. One does not simply select a coverage value in the multivariate space in the same way as one would in the univariate space.

While coverage in the multivariate space is not the same compared to the univariate space, analysts have some benefit to utilizing multivariate statistics. Specifications can either be based on a CDF contour or a constant coverage contour. Additionally, in the bivariate space, contours allow engineers a range of response points that satisfy a target level of conservatism, compared to a single response point in the univariate space. Having multiple points of the same statistical conservatism is important in case the ground test is not able to replicate some responses, but it is able to replicate others.

Numerical Demonstration

Synthesized Environments Data

For the purposes of this demonstration, consider a population of triaxial acceleration data from which a 3-variate target environment is derived. Realizations of the target environment observations consisted of zero-mean stationary random vibration 100 second time windows sampled at 8 kHz. Power spectral densities were then computed, and environments were derived. Because the power is constant with respect to frequency, one frequency point is considered for simplicity.

A correlation coefficient was selected based on experience with past test data. For X-Y, the coefficient was set to 0.3. For Y-Z, -0.3 was used. X and Z used a coefficient of 0, representing uncorrelated responses. Average PSD amplitudes were set to 0.0046, 0.0020, and 0.0005 G^2/Hz for X, Y, and Z, respectively.

To derive a MIMO environment from these synthetic data, a confidence value of 0.95 was first assumed. If a coverage of 0.9 is desired, simply computing that coverage level for each direction independently will not achieve the desired coverage for the multivariate environment. A comparison of the PSD levels determined computing based on univariate and multivariate statistics is given in Table 1. A multivariate coverage level of 0.7436 is included for comparison because that was computed to be the equivalent multivariate coverage level to using individual quantiles of 0.9 coverage. Note that "Multivariate" indicates the direct critical point method as defined by Watts et al., unless specified as the contour method from the same work [11]. Note that the derived environments levels and actual coverage of the distribution of possible acceleration responses are not equivalent for all methods.

Table 1 PSD levels computed using different statistical methods.

τ (Quantile)	Method	Critical point (G^2/Hz)		
		X	Y	Z
0.9	Univariate	5.973E-03	2.542E-03	6.339E-04
0.7436	Multivariate	7.967E-03	3.209E-03	8.047E-04
0.9000	Multivariate	8.974E-03	3.642E-03	9.162E-04
0.9000	Multivariate (Contour)	8.699E-03	3.425E-03	8.748E-04

Conclusion

This paper has described the some of the difficulties introduced in deriving specifications for MIMO, rather than single-axis, vibration tests. MIMO tests are more representative and accurate in part simply because they attempt to control more response DOF, so a simple multivariate generalization like the simultaneous prediction interval would be self-defeating. At the same time, these results have demonstrated that the correlation between variates (i.e., between different response channels in different locations and/or directions) has a significant impact on the resulting coverage level for a given CDF value. As a result, multivariate specification approaches are needed to take advantage of the full potential of MIMO vibration testing.

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