Chapter 2
Parametrising the Virtual Point Transformation and a Nonlinear Estimation Framework for Determining an Unknown Force

Julian Staiger and Frank Naets

Abstract This article examines the estimation of a force applied to a rigid interface using a model that does not explicitly consider such force. It discusses linear model-based estimation and proposes a novel approach based on a nonlinear augmented state-space model using the Virtual Point Transformation method. The proposed methodology is experimentally validated on an automotive-grade rubber mount component, showing superior performance compared to linear estimation approaches.

Keywords Virtual point transformation · Inverse load identification · Nonlinear estimation · Virtual sensing

Introduction

The problem of determining forces transmitted from sources to receivers, such as from an engine to a vehicle chassis, was historically treated by experimental Transfer Path Analysis (TPA) [1, 2] methods. In this context, the forces transmitting through connecting elements, e.g. rubber mounts, are of paramount importance due to the prominence of the structure-borne noise path in many applications. To determine inputs and outputs at inaccessible locations, Van der Seijs et al.proposed the Virtual Point Transformation (VPT) to transform a set of measured inputs and outputs [3]. However, TPA techniques are exploited in the frequency domain, compromising their usability for real-time calculations, direct use in control, and dealing with time-varying or nonlinear systems.

Recently, it has been proposed by the authors in [4, 5] to determine forces transmitting through connecting elements, such as rubber bushings, using an Augmented Kalman Filter (AKF) under the assumption of local rigid behaviour at the interfaces. This estimation is achieved by applying the VPT method to determine state-space models whose inputs are transformed into inaccessible virtual points. It makes estimating forces transmitting through connecting elements possible using measured data.

An unknown force's amplitude, position, and direction are essential in mechanical applications. Forces transmitted through connecting elements, such as rubber bushings, are crucial as they are the main pathway of structure-borne noise. It is essential to assess how effectively the bushing isolates vibrations by determining the forces transmitted through it. Furthermore, the position and direction of forces on each side of a bushing can affect its durability, leading to increased vibration, noise, or mechanical failure. Understanding these forces can also aid active control strategies employing advanced bushings. The existing literature primarily focuses on identifying unknown forces acting on mechanical systems at known locations, while forces transmitting through components or applied at unknown structure positions are less identified.

Obtaining State-Space Mdels from Experimental Data

The model-based estimation framework applied in this research utilises an experimental state-space model. The method that will be used to identify the state-space model from experimentally acquired data was originally proposed in [6]. This approach is composed of three different steps. 1) The Frequency Response Functions (FRF) of the structure of interest

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must be measured. 2) System identification algorithms, such as Polyreference least-squares complex frequency-domain (PolyMAX) [7] and Maximum Likelihood-Modal Model Method (ML-MM) [8], must be exploited to determine modal parameters from the measured set of FRFs. 3) As a final step, a complete state-space model representative of the dynamics of the structure of interest is constructed.

A complete state-space model can be defined by the in-band modal parameters (i.e. poles, mode shapes and modal participation factors) together with Residual Compensation Modes (RCM) determined from the lower and upper residual matrices to account for the out-of-band dynamic contributions as outlined by Dias et al.[9]. In this way, a complete state-space model, whose FRFs approximate the measured FRFs, can be defined as follows:

$$\dot{\tilde{\mathbf{x}}}(t) = \tilde{\mathbf{A}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{u}(t) \tag{1}$$

$$\mathbf{y}(t) = \tilde{\mathbf{C}}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{D}}\mathbf{u}(t), \tag{2}$$

where, $\mathbf{u}(t) \in \mathbb{R}^{n_i \times 1}$ is the vector of inputs while $\mathbf{y}(t) \in \mathbb{R}^{n_o \times 1}$ denotes the output vector constituted by displacements. Matrices $\tilde{\mathbf{A}} \in \mathbb{C}^{n \times n}$, $\tilde{\mathbf{B}} \in \mathbb{C}^{n \times n_i}$, $\tilde{\mathbf{C}} \in \mathbb{C}^{n_o \times n}$ and $\tilde{\mathbf{D}} \in \mathbb{C}^{n_o \times n_i}$ are the state, input, output and feedthrough matrices, respectively, with n representing the number of states and n_i,n_o the number of inputs and outputs respectively. These state-space matrices are given as follows:

$$\tilde{\mathbf{A}} = \begin{bmatrix} \mathbf{\Lambda}_{ib,RCM} & 0 \\ 0 & \mathbf{\Lambda}^H_{ib,RCM} \end{bmatrix} = \begin{bmatrix} \mathbf{\Lambda}_{ib} & 0 & 0 & & & \\ 0 & \mathbf{\Lambda}_{LR} & 0 & & \ddots & \\ 0 & 0 & \mathbf{\Lambda}_{UR} & & & \\ & & & \mathbf{\Lambda}^H_{ib} & 0 & 0 \\ & \ddots & & 0 & \mathbf{\Lambda}^H_{LR} & 0 \\ & & & 0 & 0 & \mathbf{\Lambda}^H_{UR} \end{bmatrix},$$

$$\tilde{\mathbf{B}} = \begin{bmatrix} \mathbf{L}_{ib,RCM} \\ \mathbf{L}^H_{ib,RCM} \end{bmatrix} = \begin{bmatrix} \mathbf{L}_{ib} \\ \mathbf{L}_{LR} \\ \mathbf{L}_{UR} \\ \mathbf{L}^H_{ib} \\ \mathbf{L}^H_{LR} \\ \mathbf{L}^H_{UR} \end{bmatrix}, \quad \tilde{\mathbf{C}} = \begin{bmatrix} \mathbf{\Psi}_{ib,RCM} & \mathbf{\Psi}^H_{ib,RCM} \end{bmatrix} = \begin{bmatrix} \mathbf{\Psi}_{ib} & \mathbf{\Psi}_{VP,LR} & \mathbf{\Psi}_{VP,UR} & \mathbf{\Psi}^H_{ib} & \mathbf{\Psi}^H_{VP,LR} & \mathbf{\Psi}^H_{VP,UR} \end{bmatrix} \tag{3}$$

with subscripts $\bullet_{LR}$ and $\bullet_{UR}$ denoting matrices composed of modal parameters associated with the RCMs exploited to, respectively, model the dynamic contribution of the lower and upper out-of-band modes. For a detailed discussion on the definition of these sets of RCMs, the reader is referred to [6, 9, 10]. It is also worth noticing that $\tilde{\mathbf{D}}$ is a null matrix for measured displacements [11, 4] but kept in Eq. (2) and subsequently for the sake of continuity of notation.

Joint Input-State-Position-Estimation Based on Parametrized IDMs

We consider a general geometrical reduction of a force acting on a rigid interface to a virtual point, as illustrated in Fig. 1. The contributions in the virtual point $\mathbf{m}$ can be expressed as a function of the acting force, its position and direction of action as:

$$\mathbf{m} = \bar{\mathbf{R}}^p_f(\mathbf{p}))^T f = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -r_z & r_y \\ r_z & 0 & -r_x \\ -r_y & r_x & 0 \end{bmatrix} \begin{bmatrix} e_x \\ e_y \\ e_z \end{bmatrix} f = \begin{bmatrix} e_x \\ e_y \\ e_z \\ -r_z e_y + r_y e_z \\ r_z e_x - r_x e_z \\ -r_y e_x + r_x e_y \end{bmatrix} f. \tag{4}$$

with the parameter vector given as $\mathbf{p} = \begin{bmatrix} e_x & e_y & e_z & r_x & r_y & r_z \end{bmatrix}^T$ and $\bar{\mathbf{R}}^p_f(\mathbf{p}))$ can be understood as an Parametrized Interface Deformation Mode (PIDM). In this way, the parametrised expression given in Eq. (4) can be used to obtain the following nonlinear state-space representation:

$$\dot{\tilde{\mathbf{x}}}(t,\mathbf{p}) = \tilde{\mathbf{A}}_{VP}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{B}}\mathbf{T}^T_f(\bar{\mathbf{R}}^p_f(\mathbf{p}))^T f(t), \tag{5}$$

$$\mathbf{y}(t,\mathbf{p}) = \tilde{\mathbf{C}}_{VP}\tilde{\mathbf{x}}(t) + \tilde{\mathbf{D}}\mathbf{T}^T_f(\bar{\mathbf{R}}^p_f(\mathbf{p}))^T f(t). \tag{6}$$

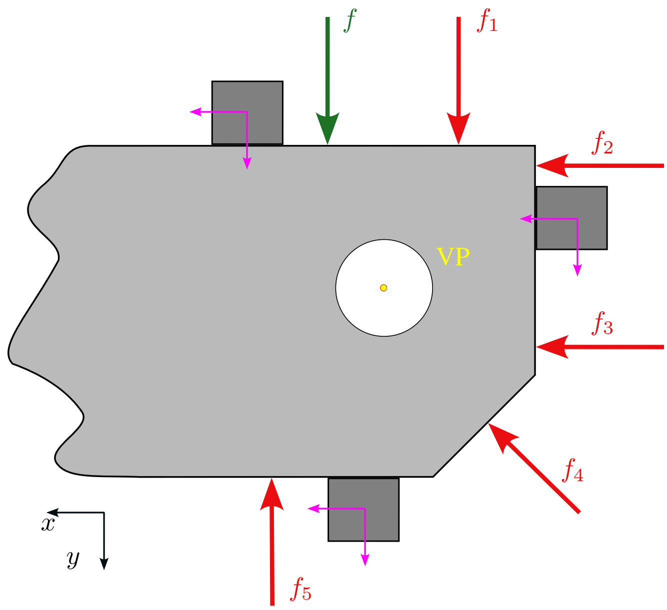


Fig. 1 Transforming the inputs of a given state-space model into a virtual point of interest and arbitrarily applied force f.

The system presented in Eqs. (5) and (6) is nonlinear due to its dependence on the parameter vector $\mathbf{p}$. At this point, to jointly estimate the parameter vector alongside the system states and the input, the state vector of the system as given in Eqs. (5) and (6) will be augmented to define the augmented model, including the dynamics states, the unknown parameters and force. To define this augmented system, the time evolution of the parameters and inputs must be modelled. A Zeroth Order Random Walk (ZORW) model will be used for both the parameters and inputs as given hereafter:

$$\dot{\mathbf{p}}(t) = \mathbf{0} + \mathbf{w}_p, \tag{7}$$

$$\dot{f}(t) = 0 + w_f, \tag{8}$$

where $\mathbf{w}_p$ and w_f are Gaussian variables with $\mathbf{w}_p \sim \mathcal{N}(0, \mathbf{Q}_p) \in \mathbb{R}^{n_p \times 1}$ and $w_f \sim \mathcal{N}(0, \mathbf{Q}_f) \in \mathbb{R}$ respectively. The deterministic-stochastic nonlinear augmented system can, thus, be denoted in short-hand notation as:

$$\dot{\tilde{\mathbf{x}}}_a(t) = \mathbf{g}(\tilde{\mathbf{x}}, \mathbf{p}, f) + \mathbf{w}_{a,NL}, \tag{9}$$

$$\mathbf{y}(t) = \mathbf{h}(\tilde{\mathbf{x}}, \mathbf{p}, f) + \mathbf{v}, \tag{10}$$

with

$$\tilde{\mathbf{x}}_a = \begin{bmatrix} \tilde{\mathbf{x}} \\ \mathbf{p} \\ f \end{bmatrix}, \quad \mathbf{g}(\tilde{\mathbf{x}}, \mathbf{p}, f) = \begin{bmatrix} \tilde{\mathbf{A}}_{VP}\tilde{\mathbf{x}} + \tilde{\mathbf{B}}_{VP}(\bar{\mathbf{R}}_f^p(\mathbf{p}))^T f \\ \mathbf{0} \\ 0 \end{bmatrix}, \quad \mathbf{w}_{a,NL} = \begin{bmatrix} \mathbf{w}_x \\ \mathbf{w}_p \\ w_f \end{bmatrix}, \quad \text{and} \quad \mathbf{h}(\tilde{\mathbf{x}}, \mathbf{p}, f) = \left[\tilde{\mathbf{C}}_{VP}\tilde{\mathbf{x}} + \tilde{\mathbf{D}}_{VP}(\bar{\mathbf{R}}_f^p(\mathbf{p}))^T f\right]. \tag{11}$$

The augmented system presented in Eqs. (9) and (10) is nonlinearly dependent on the set of unknown parameters included in $\mathbf{p}$. Linear estimation approaches like the linear Kalman filter can not be employed directly to estimate the states, inputs and parameters. Hence, a novel nonlinear Kalman filter, the Dual Augmented Extended Kalman Filter (DAEKF), introduced in [5], finds application in this article.

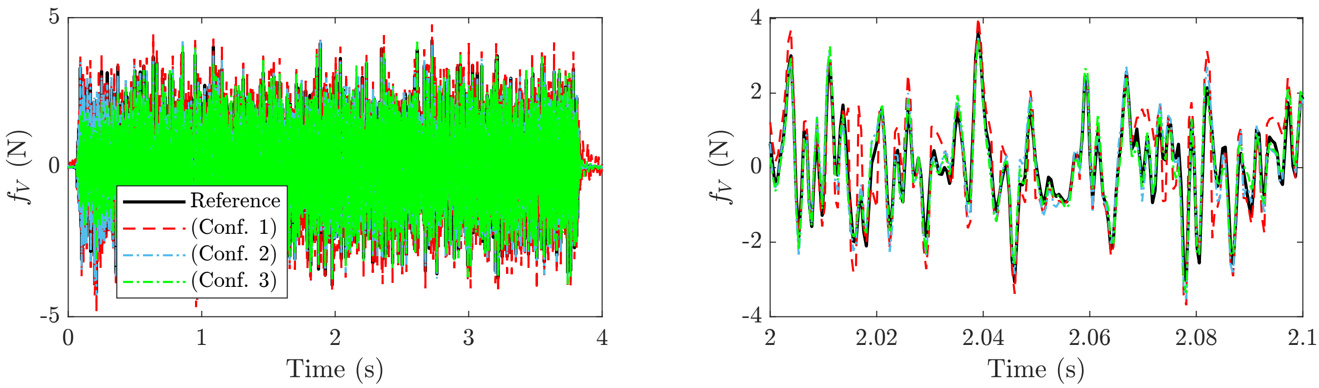
Results

The reconstruction results of a force applied by a shaker on a rigid interface attached to a rubber bushing as outlined in [5] are shown in Figure 2. Along with the estimation based on the nonlinear model introduced in Section also, two linear estimations are compared (details can be found in [5]). The three configurations which are compared are:

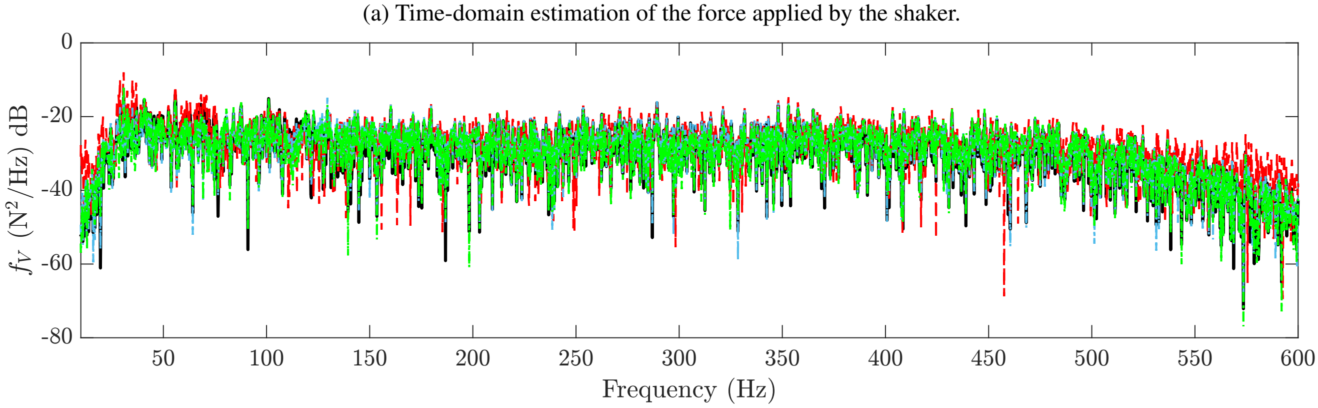
- Config.1: Linear AKF, defined with the unreduced model composed of 12 reference forces.

- Config.2: Linear AKF, defined with a virtual point reduced model estimating six virtual point contributions $\hat{\mathbf{m}}_{VP}$ belonging to the considered virtual point.
- Config.3: DAEKF estimation employing a nonlinear model as established in Eqs. (9) and (10) yielding an estimate of the force acting at the unknown position.

Analysing Fig. 2, it is evident that the applied force is best reconstructed by configuration 3. This trend is visible in the time- and frequency-domain and validates the performance of the nonlinear model and the use of the DAEKF.



(a) Time-domain estimation of the force applied by the shaker.



(b) Frequency-domain estimation of the force applied by the shaker.

Fig. 2 Time- and frequency-domain estimation results for the reference force f.

Conclusion

This paper compares three methodologies to determine an unknown force applied on a rigid interface while introducing the concept of PIDMs. Two of these approaches are based on linear AKF schemes, while the other employs a nonlinear augmented model defined via VPT by exploiting PIDMs. For estimation with the nonlinear model, the recently proposed DAEKF approach finds application in this article. The results obtained using a linear AKF based on the unreduced model showed the least accurate fit in both frequency and time domains when reconstructing the experimentally applied force. However, this was followed by the configuration using the same AKF with the virtual point reduced model. The reported results indicate that the strategy using the defined augmented nonlinear system combined with the DAEKF outperforms the previous two configurations, proving more suitable for determining an unknown force applied on a rigid interface.

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