Chapter 3
Parameter Rejection in Sensitivity-based Model Updating using Output Feedback Eigenstructure Assignment

Martin D. Ulriksen and Dionisio Bernal

Abstract Model updating involves a parameter estimation problem, in which a set of model parameters is adjusted to minimize the discrepancy between identified system features and the associated model predictions. To promote uniqueness in the estimation problem, the number of included parameters must be no larger than the number of features. This often results in the model updating being cast with a (proper) subset of the uncertain parameters, whereby the errors induced by treating the omitted parameters at their nominal values are accepted. In this paper, we review a scheme that attempts to mitigate the noted issue by minimizing the sensitivity of the parameter estimation to fluctuations of the omitted parameters around their nominal values. Control over the sensitivities is realized by adjusting output feedback gains in a closed-loop setting, which, given that the operation is offline, can be implemented through processing of open-loop input-output data. The scheme is tested in the context of a numerical example with a truss system.

Keywords Model updating · parameter estimation · parameter rejection · output feedback · eigenstructure assignment

Introduction

The parameter estimation in model updating constitutes a minimization problem, wherein the objective function is some measure of the discrepancy between identified system features and the corresponding model predictions. A common setting, which is also chosen in this study, is to let modal data comprise the features and to let the updating parameters relate to the system's mass and/or stiffness [1]. In the widely used sensitivity method, which is the one of interest here, the updating parameters are adjusted by iteratively solving a Taylor linearization [2]. The parameter estimation problem in each iteration proves ill-posed when the number of updating parameters exceeds the number of identifiable system features, which is often encountered in vibration analysis [2].

The standard procedure for mitigating the noted posedness issue in the sensitivity method is to treat some of the uncertain parameters at their nominal values and omit them from the updating. Yet, such deselection may distort the parameter estimation, so, somewhat inspired by the work on disturbance rejection in control engineering [3], we propose a parameter rejection scheme that can be directly included in the sensitivity method to (theoretically) nullify the effect of the discarded parameters. To be specific, the proposed scheme concerns design and use of closed-loop features that allow for discarding some uncertain parameters from the model updating while still rejecting their influence in the parameter estimation. This parameter rejection is attained by rendering the closed-loop features invariant to the discarded parameters by use of output feedback eigenstructure assignment [4]. The proposed scheme also allows for designing multiple closed-loop systems to increase the number of features, but since this part has already been covered in details in the literature [5], we will focus on the parameter rejection.

In most vibration applications, it suffices to cast the model updating in an offline setting instead of operating online in real time. Consequently, the output feedback that enables the eigenstructure assignment for parameter rejection can be realized *virtually* through processing of measured open-loop input-output data [6]. This virtual implementation, which is formulated

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Brian Damiano et al. (eds.), *Structural Health Monitoring & Machine Learning, Vol. 12*, Conference Proceedings of the Society for Experimental Mechanics Series, https://doi.org/10.13052/97887-438-0157-3_3

under the assumption that the considered system is linear and time-invariant (LTI) during the open-loop data collection, has several noteworthy merits. Two immediate ones are that it eliminates the practical overhead associated with utilizing physical controllers [7], and that it allows for computing several closed-loop systems—and hereby eigenstructures—based on a single open-loop input-output data realization.

Background Theory and Problem Statement

We consider a physical system $\mathcal{S}$ with known inputs $u(t) \in \mathbb{R}^r$ and outputs $y(t) \in \mathbb{R}^m$. The physics-based model representation of system $\mathcal{S}$ is an LTI state-space formulation of order $n \in 2\mathbb{N}$, namely,

$$\dot{z}(t) = Az(t) + Bu(t), \tag{1a}$$

$$y(t) = Cz(t), \tag{1b}$$

with the state vector $z(t) \in \mathbb{R}^n$ and the matrices $A \in \mathbb{R}^{n \times n}$, $B \in \mathbb{R}^{n \times r}$, and $C \in \mathbb{R}^{m \times n}$. We note that model (1) is formulated under the assumption that the direct transmission is zero or already subtracted from the outputs in (1b), which is chosen, without any loss of generality, to simplify the developments later in the paper.

Now, let the model parameters to be updated be $\theta \in \Theta \subset \mathbb{R}^p$, which, with respect to (1), implies that $A = A(\theta) \in \mathbb{R}^{n \times n}$, $B = B(\theta) \in \mathbb{R}^{n \times r}$, and $C = C(\theta) \in \mathbb{R}^{m \times n}$. These matrices follow as

$$A = \begin{bmatrix} 0 & I \\ -M^{-1}K & -M^{-1}D \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ M^{-1}B_2 \end{bmatrix}, \qquad C = \begin{bmatrix} C_\mathrm{d} - C_\mathrm{a}M^{-1}K & C_\mathrm{v} - C_\mathrm{a}M^{-1}D \end{bmatrix}, \tag{2}$$

with the mass, damping, and stiffness matrices $M = M(\theta)$, $D = D(\theta)$, $K = K(\theta) \in \mathbb{R}^{\frac{n}{2} \times \frac{n}{2}}$, the input-distribution matrix $B_2 \in \mathbb{R}^{\frac{n}{2} \times r}$, and the output-selection matrices C_d, C_v, $C_\mathrm{a} \in \mathbb{R}^{m \times \frac{n}{2}}$ for displacement (d), velocity (v), and acceleration (a) outputs.

Assumption 1 *For all $\theta \in \Theta$, model (1) is observable and controllable with $M = M^\top \succ 0$, $D = D^\top \succ 0$, and $K = K^\top \succ 0$.*

With the chosen model parametrization, we assume that system $\mathcal{S}$ is in the *unknown* configuration $\theta_* \in \Theta$. The aim is to estimate θ_* by minimizing the discrepancy between the system features and the model predictions and then adjust model (1) accordingly. Let $f_\mathcal{S} \in \mathbb{C}^q$ be the identified system features and $f_{\mathcal{M}(\theta)} \in \mathbb{C}^q$ the corresponding model predictions, then

$$\theta_* \approx \hat{\theta}_* = \underset{\theta \in \Theta}{\arg\min}\ \mathcal{H}(\theta), \tag{3}$$

where $\mathcal{H}$ is the objective function that expresses the selected discrepancy measure. Minimization problem (3) is generally nonlinear and is, by means of the sensitivity method, solved iteratively through the Taylor linearization

$$f_\mathcal{S} - f_{\mathcal{M}(\theta_k)} = \mathcal{J}_{\theta_k}\big(\theta_{k+1} - \theta_k\big). \tag{4}$$

Here, we have introduced the counter $k \in \mathbb{N}_0$ along with the Jacobian matrix $\mathcal{J}_{\theta_k} \in \mathbb{C}^{q \times p}$, which is evaluated at $\theta_k \in \Theta$ using model (1). Formulation (4), which is initialized at $k = 0$ with a *known* θ_0, constitutes an inverse problem for the unknown parameter vector θ_{k+1}. It is solved in an ℓ_2-norm sense, whereby it is assumed that $\mathrm{rank}(\mathcal{J}_{\theta_k}) = p$. A necessary condition for satisfying $\mathrm{rank}(\mathcal{J}_{\theta_k}) = p$ is obviously that $q \geq p$, and since we choose to use the imaginary part of an eigenvalue subset as features, p is bounded from above by the number of identifiable eigenvalues of system $\mathcal{S}$. Some parameters that are actually unknown and uncertain must typically be discarded in the updating to satisfy $q \geq p$. Let these parameters be gathered in $\mu \in \mathbb{R}^h$, then we seek to pose (4) such that the influence of μ is rejected without including μ explicitly in (4).

Eigenstructure Assignment for Parameter Rejection

The influence of μ is rejected in the estimation of θ after (4) if the features in f are invariant to changes in μ. Let us return to model (1) under Assumption 1 and extend it with some *fictive* inputs, $\tau(t)$, that are spatially distributed like $u(t)$ and comply with the static output feedback law

$$\tau(t) = Gy(t). \tag{5}$$

Here, $G \in \mathbb{C}^{r\times m}$ is a gain matrix, which can be complex-valued since the feedback is realized virtually. By adding the fictive inputs to model (1), we attain the resulting virtual closed-loop model

$$\dot{z}(t) = Wz(t) + Bu(t), \tag{6a}$$
$$y(t) = Cz(t), \tag{6b}$$

with the closed-loop state matrix

$$W = W(\theta) \triangleq A + BGC \in \mathbb{C}^{n\times n}. \tag{7}$$

The eigendecomposition of the closed-loop state matrix is written as $W = \Psi\Lambda\Phi$, with the jth right eigenvector $\psi_j \in \mathbb{C}^n$, the jth left eigenvector $\phi_j \in \mathbb{C}^{1\times n}$, and the jth eigenvalue $\lambda_j \in \mathbb{C}$ for $j = 1, \ldots, n$. It thus follows that $\phi_j\,(A + BGC - \lambda_j I) = 0$, which can be written in the partitioned form

$$\begin{bmatrix} A^\top - \lambda_j I & C^\top \end{bmatrix} \begin{bmatrix} \phi_j^\top \\ \gamma_j \end{bmatrix} = 0, \tag{8}$$

where we, for notational convenience, have introduced $\gamma_j \triangleq G^\top B^\top \phi_j^\top \in \mathbb{C}^m$. With respect to (8), it is opportune to note that Assumption 1 yields $\text{null}[A^\top - \lambda_j I \quad C^\top] \subset \mathbb{C}^{(n+m)\times m}$, where null denotes (right) null space. As previously outlined, the scope of the eigenstructure assignment is to render some of the closed-loop eigenvalues, λ_j, invariant to changes in μ. Lemma 1 provides some basic insight to address this scope.

Lemma 1 *Let $\lambda(W)$ be the eigenspectrum of W, let δW be a perturbation of W, and let $(\tilde{\lambda}_j, \tilde{\phi}_j)$ be a left eigenpair of $W + \delta W$. Then, $\tilde{\lambda}_j \in \lambda(W)$ if $\tilde{\phi}_j^\top \in \text{null}(\delta W^\top)$.*

Now, let δW_μ and δW_θ denote the perturbations of W induced by the perturbations $\delta\mu$ and $\delta\theta$, respectively. We then seek to design closed-loop model (6) such that $\phi_j^\top \in \text{null}(\delta W_\mu^\top)$ for some j. To this end, we return to (8) under Assumption 1 and let a basis for $\text{null}[A^\top - \lambda_j I \quad C^\top]$ be $\mathcal{Z}_j = [\mathcal{X}_j^\top \;\; \mathcal{Y}_j^\top]^\top$, with $\mathcal{X}_j \in \mathbb{C}^{n\times m}$ and $\mathcal{Y}_j \in \mathbb{C}^{m\times m}$. Hereby,

$$\phi_j^\top = \mathcal{X}_j \alpha_j, \tag{9}$$

where $\alpha_j \in \mathbb{C}^m$. If $\phi_j^\top \in \text{null}(\delta W_\mu^\top)$, then pre-multiplying (9) by $\delta W_\mu^\top$ yields

$$\delta W_\mu^\top \mathcal{X}_j \alpha_j = 0, \tag{10}$$

so a sufficient condition for the existence of an α_j that satisfies relation (10) is $\text{rank}(\delta W_\mu) < m$. Let δA_μ, δB_μ, and δC_μ be the perturbations of A, B, and C induced by $\delta\mu$. Then, $\delta W_\mu = \delta A_\mu + (B + \delta B_\mu)G(C + \delta C_\mu) - BGC$, which, when plugged into (10), results in

$$\Big(\delta A_\mu^\top + (C + \delta C_\mu)^\top G^\top (B + \delta B_\mu)^\top - C^\top G^\top B^\top\Big)\phi_j^\top = 0. \tag{11}$$

By introducing $\tilde{\gamma}_j \triangleq G^\top\big(B + \delta B_\mu\big)^\top \phi_j^\top$ and assuming that $\text{rank}\big(\delta W_\mu\big) < m$, we can combine (8) and (11) to attain the augmented system of homogeneous equations

$$\begin{bmatrix} A^\top - \lambda_j I & C^\top & 0 \\ \delta A_\mu^\top & -C^\top & C^\top + \delta C_\mu^\top \end{bmatrix} \begin{bmatrix} \phi_j^\top \\ \gamma_j \\ \tilde{\gamma}_j \end{bmatrix} = 0, \tag{12}$$

which is the eigenstructure assignment formulation for parameter rejection. As (12) implies, the spatial distribution but not the magnitude of $\delta\mu$ enters the formulation if the magnitude parameterization in μ is linear.

We consider g closed-loop eigenvalues and their associated left eigenvectors and order them as $\big(\lambda_j, \phi_j\big)$ for $j = 1, \ldots, g$. To this end, let $\Omega \triangleq B^\top\big[\phi_1^\top \;\ldots\; \phi_g^\top\big] \in \mathbb{C}^{r\times g}$ and $\Gamma \triangleq \big[\gamma_1 \;\ldots\; \gamma_g\big] \in \mathbb{C}^{m\times g}$, then we attain the identity

$$\Omega^\top G = \Gamma^\top, \tag{13}$$

which is an inverse problem for the unknown G. In the overdetermined case of $g > r$, formulation (13) will, as demonstrated in [5], typically be inconsistent, hence we confine the discussion to $g = r$.

Theorem 1 *Assume that Ω in (13) has full rank with $g = r$. Then, r eigenvalues of closed-loop model (6) can be assigned invariant to δW_μ for $\text{rank}(\delta W_\mu) < m$.*

Implementation of the Scheme

The model updating is formulated with features that are composed of the imaginary parts of q closed-loop eigenvalues assigned using (12). According to theorem 1, r eigenvalues of closed-loop model (6) can be assigned invariant to $\delta\mu$ if $\text{rank}(\delta W_\mu) < m$, but we often encounter $p > r$, where it is recalled that p is the number of parameters in θ. In such cases, we must therefore design multiple closed-loop models to satisfy $q \geq p$.

Let d be the number of closed-loop models, which must be selected such that it satisfies $q \geq p$. For $l = 1, \ldots, d$, the gain matrix $G^{(l)}$ is designed using the physics-based model in parameter configuration θ_k and with the magnitudes in $\delta\mu$ being user-defined. Hereby, (7) yields the closed-loop state matrix $W^{(l)}$, with the assigned eigenvalues $\lambda_j^{(l)}$ and left eigenvectors $\phi_j^{(l)}$ for $j = 1, \ldots, r$. The imaginary parts of the assigned eigenvalues for all d realizations, which amount to $q = dr$, are gathered to constitute the model-based feature vector, $f_{\mathcal{M}(\theta_k)}$.

For the physical system, $\mathcal{S}$, which is in parameter configuration θ_*, the state matrix estimate $\hat{A}_\mathcal{S}$, the input matrix estimate $\hat{B}_\mathcal{S}$, and the output matrix estimate $\hat{C}_\mathcal{S}$ are inferred from system identification. Subsequently, the matrix triplet is used to form the experimental closed-loop state matrix estimate

$$\hat{W}_\mathcal{S}^{(l)} = \hat{A}_\mathcal{S} + \hat{B}_\mathcal{S} G^{(l)} \hat{C}_\mathcal{S} \tag{14}$$

for each of the d gain matrices. The experimental feature vector, $f_\mathcal{S}$, is populated with the imaginary parts of the eigenvalues of the q closed-loop eigenmodes that correspond to the model-based ones.

The proposed model updating scheme, which follows formulation (4), is terminated when/if convergence is attained according to the simple criterion

$$\epsilon_f \geq \max \left| f_\mathcal{S} - f_{\mathcal{M}(\theta_k)} \right|, \tag{15}$$

where ϵ_f is a user-defined threshold, which must satisfy $0 < \epsilon_f < \max |f_\mathcal{S} - f_{\mathcal{M}(\theta_0)}|$. It is opportune to note that the gain design must be conducted in each iteration to satisfy the invariance property in the features.

Numerical Example

We consider the two-dimensional finite-element truss system depicted in Fig. 1, with known inputs at both degrees of freedom (DOF) in node 8 and known displacement outputs at both DOF in nodes 4, 9, and 11. We let θ be composed of mass perturbations in nodes 2, 4, and 6 and let μ be the spring stiffnesses k_1 and k_2. The term *nominal model* refers to the updating model and *truth model* to the model replacing the physical system, $\mathcal{S}$. In the nominal model, the bars have a cross-sectional area of 10^{-4} m^2, a modulus of elasticity of 200 GPa, and a mass density of $8,000$ kg/m^3, so the mass of the truss is approximately 300 kg. The nominal model is classically damped with 2 % damping in each eigenmode, and both springs have the stiffness 100 kN/m. The mass matrix of the truth model is taken as the mass matrix of the nominal model with the addition of mass perturbations of 5 kg in node 2, 8 kg in node 4, and 3 kg in node 6. The springs in the truth model have the stiffnesses $k_1 = 70$ kN/m and $k_2 = 150$ kN/m, and the moduli of elasticity in the truth model are assigned random

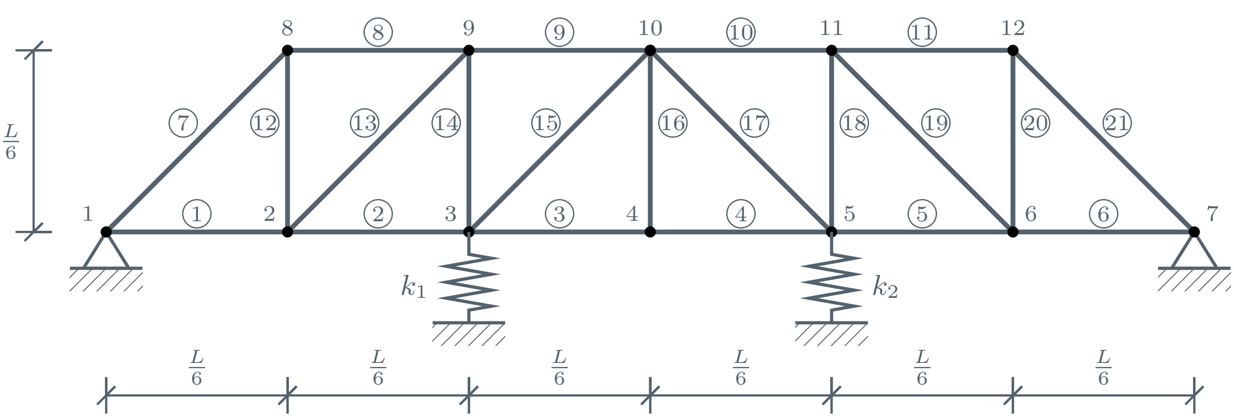


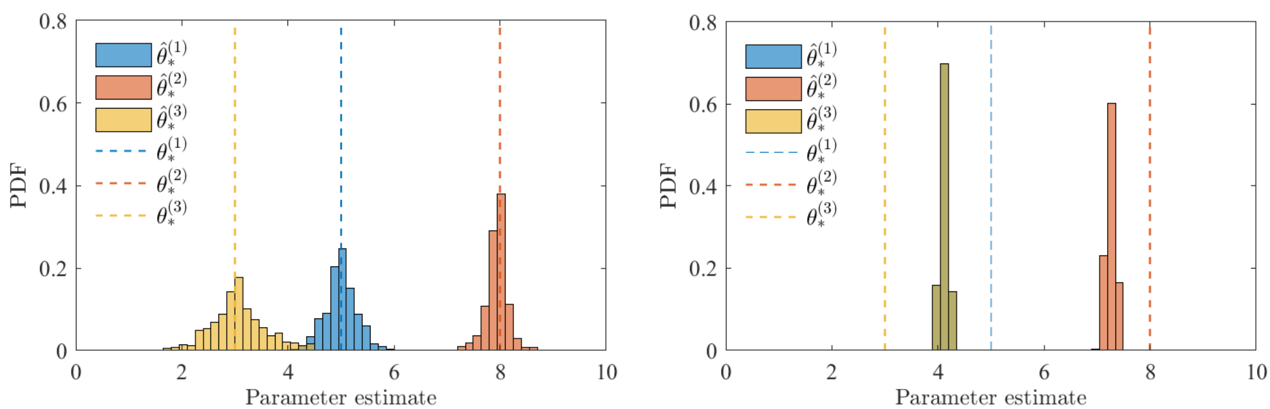
Fig. 1 Finite-element truss system with node and element numbering

perturbations that allow for ± 1 % deviation with respect to the nominal value. The truth model has the same damping matrix as the nominal model.

The truth model is simulated in a Monte Carlo setting with 500 realizations, where the inputs excite the first four eigenmodes of the model, which have the damped eigenfrequencies $\{10.34, 17.47, 19.50, 24.29\}$ Hz. The inputs and outputs are temporally discretized with a sampling frequency of 200 Hz for $100,000$ samples and subsequently contaminated with 5 % additive white Gaussian noise. The open-loop estimates $\hat{A}_{\mathcal{S}}$, $\hat{B}_{\mathcal{S}}$, and $\hat{C}_{\mathcal{S}}$ are inferred via a data-driven subspace identification algorithm [8] with a model order of 20. The identified matrices are subsequently modally truncated to contain only the first four eigenmodes and their complex conjugates.

The setting $r = 2$ and $m = 6$ allows for designing gain matrices that each assigns two eigenvalues invariant to a perturbation of rank 5 or less, and we note that $\mathrm{rank}(\delta W_\mu) = 2$. With three unknown parameters to be estimated and four open-loop eigenmodes that are identifiable, we choose to operate with two different gain matrices and assign two eigenvalues with each of them. The first gain matrix assigns the first and second closed-loop eigenvalues at 0.95 times the first and second open-loop eigenvalues of the nominal model, while the second gain matrix assigns the third and fourth closed-loop eigenvalues at 0.95 times the third and fourth open-loop eigenvalues of the nominal model.

The parameter estimation is initialized with $\theta_0 = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix}^\top$, and we recall that the true configuration is $\theta_* = \begin{bmatrix} 5 & 8 & 3 \end{bmatrix}^\top$. Fig. 2 depicts the Monte Carlo simulation-based probability density functions (PDFs) attained for the parameter estimates $\hat{\theta}_*$ with and without parameter rejection. Evidently, the rejection of μ improves the estimation accuracy, with the means of $\hat{\theta}_*^{(1)}$, $\hat{\theta}_*^{(2)}$, and $\hat{\theta}_*^{(3)}$ deviating, respectively, 0.2 %, 0.63 %, and 2.33 % from the true values. The corresponding errors without the parameter rejection are 18 %, 9 %, and 36.67 %. It can also be seen that the parameter rejection reduces the estimation precision, which is expected since the scheme without rejection employs only $\hat{A}_{\mathcal{S}}$ (and the associated errors) from the truth model.



(a) (b)

Fig. 2 PDFs for the parameter estimates $\hat{\theta}_*^{(1)}$, $\hat{\theta}_*^{(2)}$, and $\hat{\theta}_*^{(3)}$ based on 500 realizations; (a) with parameter rejection and (b) without parameter rejection (where $\hat{\theta}_*^{(1)}$ and $\hat{\theta}_*^{(3)}$ are equal).

Closing Remarks

The paper reviews a scheme that facilitates parameter rejection in sensitivity-based updating of LTI models by means of eigenstructure assignment. The premise of the scheme is to conduct the updating with closed-loop eigenvalue features that are assigned theoretically invariant to the parameters to be rejected. The invariance formulation calls for the spatial distribution of the parameters in question but not the associated magnitudes. Each gain allows for rendering r eigenvalue features invariant to parameters that induce perturbations in the resulting closed-loop state matrix of rank $m-1$ or less, where r and m are the number of inputs and the number of outputs. Compared to sensitivity-based model updating without parameter rejection, the proposed scheme comes with the price of added computational complexity due to the required eigenstructure assignment.

Acknowledgments The work conveyed in this paper was conducted by the authors during the second author's Otto Moensted Visiting Professorship at Aarhus University, Denmark. The financial support from the Otto Moensted Foundation is gratefully acknowledged.

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