Chapter 8
Analytical Generalization of the T and A-theta Integrals for the Study of Cracking in Three-dimensional Orthotropic Medium

Loïc Chrislin Nguedjio, Rostand Moutou Pitti, Benoit Blaysat, Pierre Kisito Talla, Fréderic Dubois, and Naman Recho

Abstract An analysis of the fracture phenomenon in wood shows that it is significantly influenced not only by mechanical loads and the heterogeneous nature of this material but also by multiple parameters related to thermal and moisture variations. It becomes important to propose predictive models that account for these climatic parameters in order to better assess the risks of failure and ensure the safety of wooden structures. In this paper, a new mathematical formulation of the invariant integral, **T** and **A**-theta is proposed to evaluate the energy release rate during the determining od fracture toughness. The formulation is extended to the three-dimensional case and is based on a generalization of Noether's theorem as a free energy that includes both real and virtual fields for mechanical and moisture loads. The resulting model allows for a decoupling of mixed fracture modes and the analysis of the impact of orthotropy and wood shrinkage-swelling on cracking parameters. The independence of the integration contour is then verified, and a finite element procedure is implemented using the Cast3M software to validate the proposed model.

Keywords Fracture · Path-independent integral · Moisture fields · Orthotropic materials · Three-dimensional modelling

Introduction

In the diagnosis and maintenance of civil engineering structures, crack detection is a crucial aspect to ensure the safety of both the structures and their users. This scientific field presents several challenges, particularly for wooden structures, due to the heterogeneous and anisotropic nature of wood, whose overall behavior is significantly influenced by variations in humidity and temperature [1]. Fracture mechanics provides a framework for analyzing the behavior of materials containing cracks and for determining the conditions necessary for crack initiation, propagation, and arrest. It is generally based on two approaches: the local approach and the global approach. The local approach relies on the formulation of continuous stress and displacement fields, which are disturbed by singularities at the crack tip [2]. The global or energetic approach is based on the energy balance of the structure, enabling the isolation of the energy release rate required for the creation of new crack surfaces. To quantify this energy while overcoming singularity issues, invariant integrals appear to be a promising alternative under certain assumptions, as demonstrated by numerous studies in the literature [3, 4].

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Among the most commonly used invariant integrals, Rice's **J** integral [5] allows for the calculation of the energy release rate in an elastic cracked medium under pure opening or shear fracture modes. However, in the case of wood, which is often subject to mixed-mode cracking, this integral presents limitations, necessitating the use of another integral, namely the **M** integral [6]. Moreover, these classical integrals are restricted to two-dimensional problems and do not take into account the three-dimensional effects, which are essential when dealing with an orthotropic material such as wood [7]. Beyond the consideration of three-dimensional effects, it is also necessary to extend the modeling approach to variable environments [1]. The first studies on crack modeling in wood using invariant integrals in variable climatic conditions date back to the work of Moutou Pitti et al. [8], further developed by Hassen et al. [1] and Seif et al. [9] in their study of temperature effects through the **A** integral.

This paper focuses on modeling the effect of humidity and incorporating a moisture gradient into the analytical formulation of the **A** integral. The mathematical analyses are revisited, leading to a new expression of the **A** integral, first in 2D and then extended to 3D. A numerical implementation scheme is proposed, along with results illustrating the influence of humidity on the energy release rate.

Background

The formulation of the **T** integral is derived from Noether's theorem [10] and a combination of real and virtual mechanical fields, extended to kinematically admissible moisture content variations. The system's free energy F* is then expressed in a form that incorporates real and virtual displacement fields, respectively $(u,\ v)$, real stress fields ${\sigma_{ij}}^{u}$ and virtual stress fields ${\sigma_{ij}}^{v}$, as well as the moisture content field H^{u}, considered only in a real configuration, all of which remain kinematically admissible. By applying Noether's theorem and imposing the total Lagrangian null condition $\left(\delta L = \int_V \int_t \delta F^* dV dt = 0\right)$, successive integrations lead to the following expression for the **A** integral, established to the same steps as Moutou Pitti et al. [8]:

$$A\theta^{3D} = -\frac{1}{2}\int_V \left[\sigma_{ij,k}^{(v)}\cdot u_i - \sigma_{ij}^{(u)} v_{i,k} - H_{,k}^{(u)}\cdot v_i + H^{(u)}\cdot v_{i,k}\right]\cdot\theta_{k,j}\cdot dV - \frac{1}{2}\int_{S_{CF_+}}\left[\left(\sigma_{ij}^{(v)} u_{i,k} + \left(\sigma_{ij}^{(u)} - H^{(u)}\right) v_{i,k}\right)\right]\theta_k\cdot dS$$
$$+\frac{1}{2}\int_V\left[\left(\sigma_{ij}^{(v)} u_{i,k}\right)_{,j} + \left(\left(\sigma_{ij}^{(u)} - H^{(u)}\right) v_{i,k}\right)_{,j} - \sigma_{ij}^{(v)}\left(u_{i,j}\right)_{,k} - \left(\sigma_{ij}^{(u)} - H^{(u)}\right)\left(v_{i,j}\right)_{,k} + H_{,k}^{(u)} v_{i,j} - H_{,j}^{(u)}\cdot v_{i,k}\right]\theta_k\cdot dV \tag{1}$$

θ is a field with specific properties designed to circumvent boundary issues in finite element calculations. It is equal to one within the integration contour V and zero in the external domain while remaining continuously differentiable within V (Figure 1). The integral is divided into three main terms, with an extension to steady-state behavior represented by the highlighted quantity. The first term corresponds to the energy required for stationary crack growth while accounting for hygroscopic effects and is used for mixed-mode decoupling. The second term introduces a pressure effect on the crack lips, while the third term facilitates the separation of mixed modes during crack propagation.

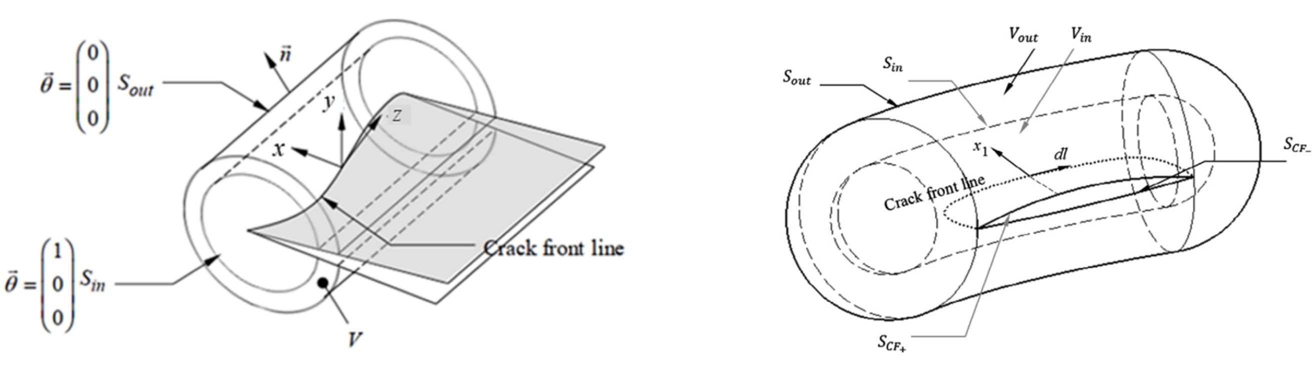

Fig. 1 θ field in 3D (left), Integration domain (right).

The particularity of this work lies in the implementation of the coupling between mechanical and moisture fields. The first term of the previous invariant integral, associated with stationary crack behavior, can be reformulated as follows:

$$\boldsymbol{A\theta^{3D}} = -\frac{1}{2}\int_V \left[{\sigma_{ij,k}}^{(v)}.u_i - {\sigma_{ij}}^{(u)} v_{i,k} - \gamma_{ij}\left(\delta w\left(v_{i,j} - \emptyset_{,j}\right) - (\delta w)_{,j}\left(v_{ij} - \emptyset\right)\right)\right].\theta_{k,j}.dV \tag{2}$$

In this new formulation, γ_{ij} is the tensor responsible for coupling the material's mechanical behavior with its hygroscopic effects. The components of this tensor are derived from the Navier equation in continuum mechanics and are presented in this paper in the form of Equation 3.

$$\gamma_{ij} = \sum_{kl} C_{ijkl}.\alpha_{kl} \tag{3}$$

The compliance tensor C_{ijkl}, which characterizes the material state at a given time under a specific mechanical load, is reorganized to retain only the components relevant to the principal orthotropic directions (L, R, and T). The indices k and l are projected onto the directions of the diagonal matrix α_{kl}, which represents the tensor whose components correspond to the coefficients of the hygroscopic shrinkage-swelling model of wood, taken as the principal orthotropic reference. The quantity $\delta w = w - w_{ref}$ denotes the moisture content variation within the material relative to a reference state w_{ref}, while $(\delta w)_{,j}$ represents its gradient in the considered direction.

The virtual mechanical fields along the crack front, as well as the components of the matrix $\emptyset$, are derived from the formulations of Sih [11]. For a two-dimensional problem involving an elastic anisotropic material, the virtual displacement components v_1 and v_2 have been redefined to incorporate the behavioral changes of wood induced by the presence of an external moisture load. Their revised expressions are presented as follows:

$$\begin{aligned} v_1 = {} & K_I\sqrt{\frac{2r}{\pi}}\Re\left[\frac{1}{\mu_1-\mu_2}\left(\mu_1 p_2\sqrt{cos\varphi+\mu_2 sin\varphi}-\mu_2 p_1\sqrt{cos\varphi+\mu_1 sin\varphi}\right)\right] \\ & + K_{II}\sqrt{\frac{2r}{\pi}}\Re\left[\frac{1}{\mu_1-\mu_2}\left(p_2\sqrt{cos\varphi+\mu_2 sin\varphi}-p_1\sqrt{cos\varphi+\mu_1 sin\varphi}\right)\right] \end{aligned} \tag{4}$$

$$\begin{aligned} v_2 = {} & K_I\sqrt{\frac{2r}{\pi}}\Re\left[\frac{1}{\mu_1-\mu_2}\left(\mu_1 q_2\sqrt{cos\varphi+\mu_2 sin\varphi}-\mu_2 q_1\sqrt{cos\varphi+\mu_1 sin\varphi}\right)\right] \\ & + K_{II}\sqrt{\frac{2r}{\pi}}\Re\left[\frac{1}{\mu_1-\mu_2}\left(q_2\sqrt{cos\varphi+\mu_2 sin\varphi}-q_1\sqrt{cos\varphi+\mu_1 sin\varphi}\right)\right] \end{aligned} \tag{5}$$

where (r, φ) denotes the polar coordinates of a point P_n near the crack tip, $\Re(.)$ the real part of the complex quantity in parentheses. K_I and K_{II} are the stress intensity factor in mode I (opening crack) and mode II (shear crack), respectively. μ_1 and μ_2 are the roots of the characteristic equation, which depends on the coefficients of the modified compliance matrix. In the case of an elastic anisotropic material, this matrix takes the following general form:

$$\xi_{11}\mu^4 - 2\xi_{16}\mu^3 + (2\xi_{12} + \xi_{66})\mu^2 - 2\xi_{26}\mu + \xi_{22} = 0 \tag{6}$$

The components ξ_{ij} at a given humidity state w are expressed in terms of the components C_{ij} of the compliance matrix, taken as a reference state at humidity w_{ref}. The parameters p_i and q_i are defined as follows:

$$p_i = \xi_{11}\mu_i^2 + \xi_{12} - \xi_{16}\mu_i \text{ and } q_i = \xi_{12}\mu_i - \xi_{26} + \frac{\xi_{22}}{\mu_i} \tag{7}$$

Analysis

The computation of the stress intensity factor and the energy release rate implies a numerical implementation of the integral in Equation (2). In this study, finite element analyses are carried out using the Cast3M software. The mesh geometry (Figure 2a) corresponds to a MMCG (Mixed Mode Crack Growth) specimen, selected for its ability to separate mixed fracture modes while maintaining a significant stability zone [12]. An initial crack length of 25 mm is considered, and the calculations are performed under a plane stress configuration in pure opening mode. Figure 2b illustrates the deformed mesh after the application of both mechanical and moisture loading.

The material studied is Scots pine (*Pinus sylvestris*), with its mechanical properties at a reference humidity of 12% listed in Table 1. These properties are subsequently adjusted at each step of the finite element analysis using the Guitard formulas [13], allowing for the consideration of mechanical behavior variations due to humidity changes.

The finite element analysis must account for the stress induced by the external moisture loading, which need to be subtracted from the initial mechanical loading. This process is outlined in the algorithmic diagram shown in Figure 3.

In the absence of experimental data, the reliability of the invariant integral implementation is typically validated by ensuring its independence from the integration domain. This validation involves computing the invariant integral across

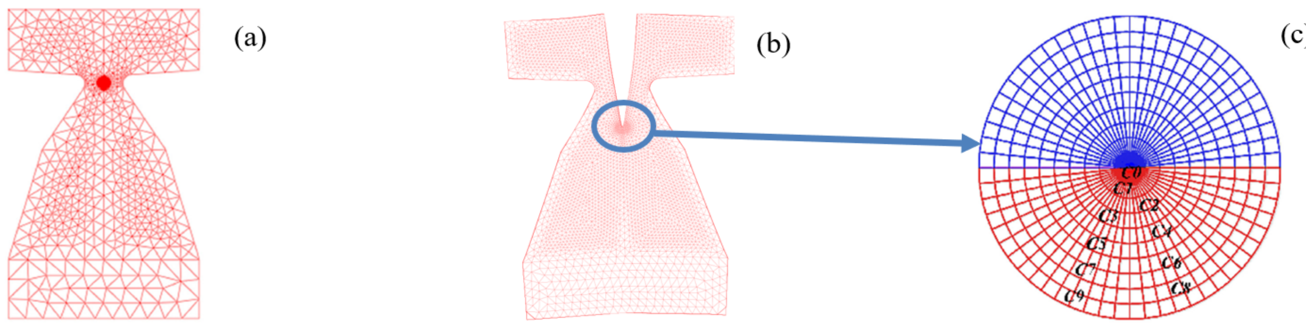

Fig. 2 (a) MMCG mesh, (b) Opening mode, (c) Radiating mesh at the crack tip.

Table 1 Properties of *Pinus sylvestris* at 12% of moisture content.

Properties	E_L (Mpa)	E_R (Mpa)	G_{RL} (Mpa)	ν_{RL}	α_L	α_R
Value	15000	600	700	0.5	0.0012	0.037

multiple rings within the integration domain. Here, the domain remains continuous through the field θ, provided that all calculations stay within its bounds neither too close to the crack tip nor too far toward the outer edge. In this study, the integration domain around the crack tip is divided into 10 rings, as illustrated in Figure 2c. Figures 4a and 4b present the calculation of the energy release rate for rings 2 to 8 under different moisture content levels.

First, a singularity is observed in the second ring of the integration domain. This is a well-known issue, as the calculations rely on stress fields that are too close to the singular region, where interpolations using asymptotic field approximations are performed with a limited number of finite elements, reducing interpolation accuracy. Beyond this, the independence of the integral is verified for rings 3 to 7. However, a disturbance reappears from ring 8 onward, as it is near the boundary of the integration domain, making the integration sensitive to boundary conditions. The fluctuation in independence ranges from

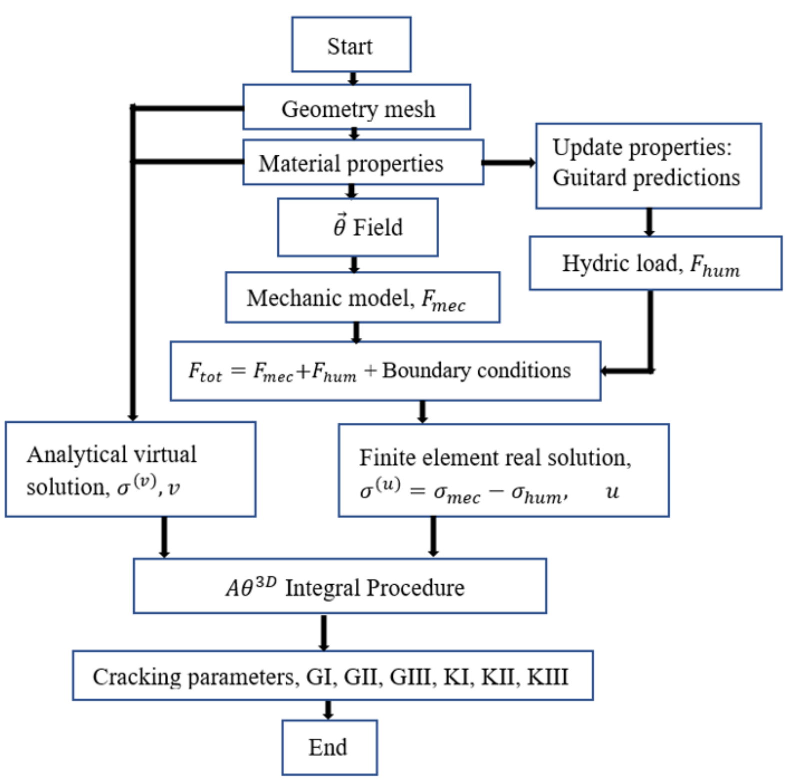

Fig. 3 Finite element numerical scheme implementation.

2% to 5%, compared to only 1% for the invariant integral $M\theta$ (equivalent to $A\theta$ in the absence of moisture loading). This suggests that the invariant integral is slightly affected when an external load is applied. Similar findings have been reported in previous studies on the independence of this integral under thermal loading [1, 9].

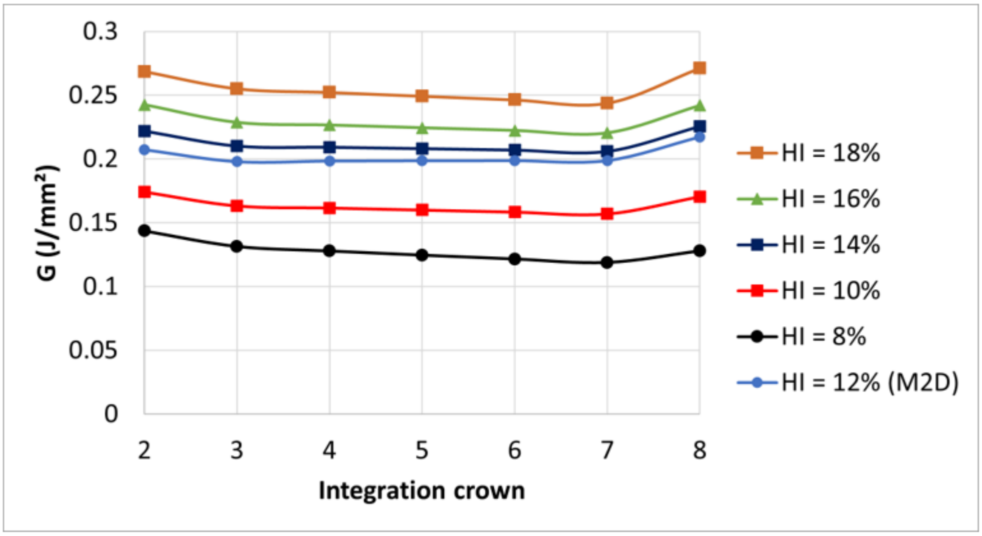

Fig. 4a Path independence verification of energy release rate for 8% to 18% moisture content.

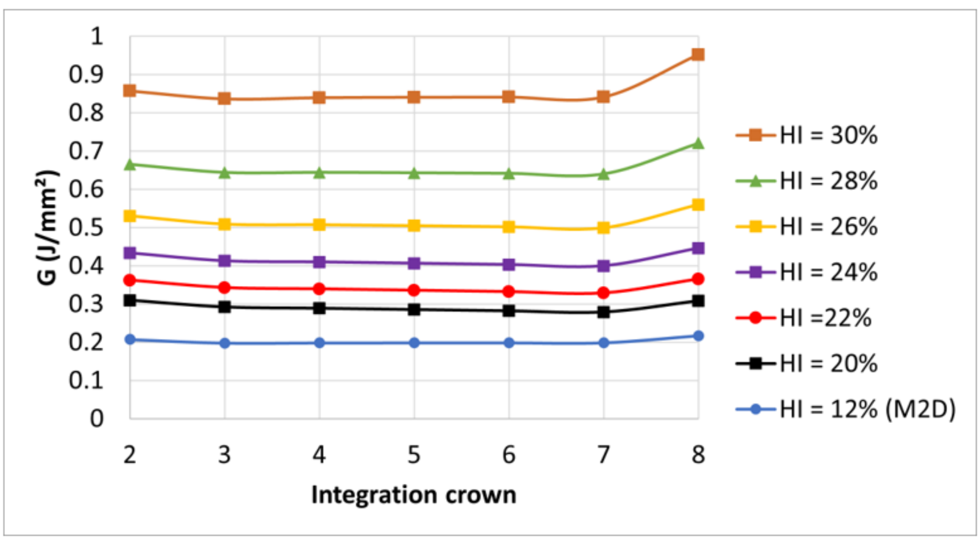

Fig. 4b Path independence verification of energy release rate for 20% to 30% moisture content.

Conclusion

This work presents a new analytical formulation of the invariant integral **T** to investigate crack propagation in wood under external hydric loading. The approach is based on Noether's theorem, with mechanical fields reformulated to incorporate hydric parameters and extended to a three-dimensional framework. As an initial validation, the first term of the integral is numerically implemented in mode I fracture for a two-dimensional MMCG wood specimen. The crack tip mesh is divided into ten concentric surface rings within the integration domain, enabling an assessment of the integration contour's independence. The computed energy release rate for each ring confirms this independence for rings 3 to 7, within a humidity range of 8% to 30%. However, numerical disturbances arise near the crack tip (rings 1 and 2) and at the outer boundary of the integration domain. Since these results assume a uniform humidity distribution within the material, future studies will examine the effects of moisture transport, mixed-mode fracture, and crack lips submitted to pressure.

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